IMAGINARY QUANTITIES:

THEIR GEOMETRICAL INTERPRETATION.

Translated from the French of M. Argand

BY

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PREFACE.

THE work now republished* is of that small number which mark an epoch in the history of science. In this short treatise is found the germ of the true theory of so-called *imaginary* quantities. Although generally attributed to the genius of Gauss, this theory was not pointed out by that great geometer until twenty-five years after the publication of Argand's work,† and it had been meanwhile re-discovered several times in both France and England. On this point we can cite no testimony more convincing than that of a German geometer, whose recent death is deplored by science. Says Hankel,‡ "the first to show how to represent the imaginary forms $A+Bi$ by points in a plane, and to give rules for

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* 1st edition, Paris. Duminil-Lesueur, 1806.

† Anzeige zur "*Theoria residuorum biquadraticum Commentatio secunda*," 1831 (Gauss Werke, t. II, p. 174).

‡ *Vorlesungen uber die complexen Zahlen und ihre functionen.* (Leipzig, 1867, p. 82).

their geometric addition and multiplication, was Argand, who established his theory in a pamphlet printed in Paris, in 1806, under the title '*Essai sur une manière de representer les quantités imaginaires dans les constructions géométriques.*' Yet this paper did not meet with public recognition until after the insertion of a note by J. F. Francais, in the *Annales de Gergonne*, Vol. IV, 1813, 1814, p. 61, in which, at the same time, Argand* also published two articles. In these articles the subject was so exhaustively treated that nothing new has since been found to add to them, and, unless some older work is discovered, Argand must be regarded as the true founder of the theory of complex quantities in a plane.

". . . . . In 1831, Gauss† developed the same idea, as is well known; but, however great his merit, as bringing this idea to the notice of science, it is none the less impossible to claim for him priority."

* Vol. IV, p. 133, and Vol. V, p. 197.

† Works, Vol. II, p. 174.

From this accurate historical *résumé*, it is seen that the work of Argand remained almost wholly unknown, having been distributed but to few persons, and not put in general circulation. Seven years later, Francais, an artillery officer at Metz, sent to the Editor of the *Annales* the outline of a theory whose germ he had found in a letter written to his brother by Legendre, the latter having obtained it from another author whose name he did not give. This article came to the notice of Argand, who immediately wrote Gergonne a note in which he made himself known as the author of the work cited in Legendre's letter, and in which he gave quite a complete summary of his pamphlet of 1806. This double publication gave rise to a discussion in the *Annales*, in which Francais, Gergonne and Servois took part, closing with a remarkable article, in which Argand explained more satisfactorily certain points in his theory, especially his demonstration of the fundamental proposition of the theory of algebraic equations, the simplest yet given, which subsequently Cauchy only reproduced, in a purely analytic, but less striking, form. These various articles, the natural sequel to Argand's pamphlet, published in a *Recueil* now very rare, are collected in an appendix to this volume. Notwithstanding their appearance in a scientific journal so well known, the views of Argand were wholly unnoticed, as appears from the fact that twenty-two years after the publication of the essay they were re-stated both by Warren, in England, and Mourey, in France, apparently without any knowledge on their part of their earlier exposition. Nor did they themselves succeed in attracting the attention of geometers, although the researches of Mourey were given in the *Lecons d'Algèbre* by Lefébure de Fourcy, and two articles, supplementary to his first work, had been published by Warren in the Philosophical Transactions. Only after Gauss had spoken, were these views taken up in Germany. They soon became familiar

to English geometers, and were the starting point of Hamilton's theory of Quaternions, while, in Italy, Bellavitis made them the basis of his *Méthode des Équipollences.** In France, Argand's theory was worked over without material addition, till its adoption by Cauchy, who expounded it in his *Exercices d'Analyse et de Physique mathématique*,† with a complete historical notice rendering Argand full justice.

In the work of this modest savant of Geneva is to be found the origin of many subsequent researches, some of which have thrown unexpected light both upon the mystery which has so long enveloped negative and imaginary quantities, as well as upon the general theory of functions, by affording a definite geometrical interpretation. Others, as yet of less importance, but perhaps destined in the future to render great services, have resulted in the creation of

* *Exposition de la Méthode des Equipollences Guisto Bellavitis.* Traduit de l'Italien par C. A. Laisant, Paris: Gauthier-Villars, 1874.

† Vol. IV, p. 157.

new methods in analytical geometry, among which may be cited those of Möbius, Bellavitis, Hamilton and Grassman. Unable to avoid the constant presence of negative and imaginary quantities in the results of analysis, or to surrender the important advantages following the use of their corresponding symbols, mathematicians had for a long time been content to employ them without fully accounting for their true nature, regarding them as signs of operations which in themselves had no meaning, yet which, under certain rules, led surely and directly, though in an obscure and mysterious manner, to results which other quantities would not have yielded, except indeed by long and difficult processes, involving the discussion of an indefinite number of particular cases. It is at last seen, however, that the impossibility of negative quantities is, in general, only apparent, and results from a generalization of the idea of quantity without any modification of the correponding analytical operations. An

analogous case is found in the very elements of arithmetic, which, however, has given rise to no difficulty. The operation of division cannot be exactly performed if we are restricted to whole numbers. But if unity be divided into equal fractions, the division is always possible, and the result becomes a *complex* expression, consisting of two numbers, one indicating multiplication, the other division. Hence arises a new class of quantities, fractions, subject to operations to which are applied the same names given to the operations on integers, which they include as particular cases. But the definitions of multiplication and division have been therefore carefully modified, to render them applicable to the new quantities. By proceeding in an analogous manner in addition and subtraction, the meaning of a negative quantity has been definitely fixed. So long as the problem is restricted to the simple determination of magnitude, the subtraction $a-b$ is impossible and absurd, if $b>a$. But if, instead of a series of magnitudes, extending from zero in a single direction, we are concerned with a series extending indefinitely in two opposite directions, and if we call *addition* an operation which consists in starting from a certain quantity in one of these two directions, and *subtraction* an inverse operation, consisting of motion in the opposite direction,—thus defined, both operations will be always possible and their results as real as those of a purely arithmetical addition.

To represent these results in a simple manner, we are led to write before the symbol, representing any quantity, a sign indicating the direction in which it is estimated. Such is the true meaning of negative quantities.

This extension of the meaning of quantity and of the operations to which it is subjected, may be carried still further. But this further representation of quantity makes the use of a geometrical notation, which, within the limits of its application, is the most luminous and

complete of all, almost indispensable. Suppose the quantity sought be subject to two causes of variation, and to depend upon two magnitudes which can be represented by any two co-ordinates fixing the position of a point in a plane. The operation of extracting the square root, for example, in the preceding case of a single variable co-ordinate, was possible only when the quantity so operated upon was of the kind denoted by plus unity. So long as $\sqrt{a}$ corresponds to the construction of a mean proportional between a and $+1$, $\sqrt{-b^2}$ indicates an impossible operation, and no point of the locus corresponding to a single variable co-ordinate can represent this result. But if both co-ordinates are made variable and the restriction to a single line be abandoned, and the definition of the square root be modified, the case is otherwise. The quantities considered do not then depend upon a single magnitude, but on two, and are for this reason called complex quantities. In operating on such a quantity, both of the quantities on which it depends are affected, exactly as in operating on a fraction we affect its two terms. Thanks to the introduction of both new quantities and new definitions of operations, $\sqrt{-b^2}$ no longer indicates an impossible operation, and the term *imaginary* is no more applicable to such a result than to fractions or negative quantities. Such is the fundamental and immediate consequence of Argand's conception. Symbols of the form $a+b\sqrt{-1}$, to which all analytical results have been reduced, are no longer either impossible or incomprehensible; they are a system of two numbers a and b, which are combined with each other just as are the co-ordinates of a point in a plane. Thenceforth, the brilliant results of the powerful analysis of Cauchy were to be translated into a geometrical language, speaking to the eyes, and the discussion of formulæ became a simple problem of the Geometry of Position, subsequently completely solved by Riemann. The theory of complex quantities which, by

the discoveries of Cauchy, had become the basis of the theory of functions, thus received at the same time a new confirmation, placing them beyond all the doubt and objections to which they had been before exposed. Such are the eminent services rendered by the discovery of Argand both to Analysis and the Philosophy of Mathematics.

But geometry, as well as analysis, though to a less degree, has profited by the introduction of these conceptions, founded on the discovery of a new bond between these two branches of the science. In Argand's work are found the beginnings of a very general method of plane analytical geometry, developed later by M. Bellavitis with great success, furnishing a uniform process for the discussion both of problems in elementary geometry and the more advanced theory of curves. The advantage of this method consists in the introduction into the calculations of the points themselves instead of their coordinates, and the consequent choice at the last moment of the most convenient system of reference. Argand was less successful in his attempts to extend his method of representing points to space of three dimensions. Indeed, this problem involved difficulties far greater than those which he had just overcome, and not till after thirty years did Hamilton at last surmount them.

We should have taken great pleasure in giving our readers some information relative to the author himself of this important tract. With this in view, we applied to M. R. Wolf, as more thoroughly acquainted with the history of science in Switzerland than any one else, and to whom we are indebted for a biographical collection, as remarkable for its profound learning as for its attractive style. M. Wolf at once kindly caused inquiries to be made in Geneva, Argand's native city. Unfortunately, the information he obtained, through Prof. Alfred Gautier, is contained in a few brief lines here cited: "I readily found the registry of birth, on July 22d, 1768, of Jean-

Robert Argand, son of Jacques Argand and Ève Canac, very probably the author of the mathematical paper in question. I learn from one who knew his family that he was for a long time a book-keeper at Paris, and I presume that he died there. He was not a near relation of Aimé Argand, * and pernaps not of the same family. He had one son who also resided in Paris." M. Wolf subsequently learned that Argand also had a daughter named Jeanne-Francoise-Dorothée-Marie-Elizabeth, married to Félix Bousquet, with whom she went to Stuttgart, where he had obtained some unimportant situation. If we add to this that, about 1813, Argand lived at Paris, *rue de Gentilly, No.* 12, as indicated in his own handwriting on the cover of the copy sent to Gergonne, we shall have stated all we have been able to learn of this original man, whose modest life will remain unknown, but whose services to

* A friend and associate of the brothers Montgolfier, who invented the lamp of that name. (1755–1803.)

science Hamilton and Cauchy have deemed worthy the gratitude of posterity.

J. HOUEL.

IMAGINARY QUANTITIES:

THEIR

Geometrical Interpretation.*

1. Let a be any arbitrary quantity. If to this quantity another equal to it be added, we may express the resulting sum by $2a$. If we repeat this operation, the result will be $3a$, and so on. We thus obtain the series $a, 2a, 3a, 4a, \ldots$, each term of which is derived from the preceding by the same operation, capable of indefinite repetition. Let us consider the series in reverse order, namely, $\ldots, 4a, 3a, 2a, a$. As before, each term of this new series may be regarded as derived from the preceding by an operation which is the reverse of the

* Essay on the Geometrical Interpretation of Imaginary Quantities, by R. Argand. Second edition with preface by M. J. Hoüel, and extracts from the *Annales de Gergonne*, Paris, Gauthier-Villars, 1874. From the French, by Prof. A. S. Hardy, Dartmouth College.

former; yet, between these series there is this difference: the first may be indefinitely extended, but the second cannot. After the term a, we should obtain 0, but beyond this point the quantity a must be of such a nature as to permit our operating on zero as we did on the other terms . . . ., $4a$, $3a$, $2a$, a. But this is not always possible. If, for example, a represents a material weight, as a gram, the series . . . ., $4a$, $3a$, $2a$, a, o, cannot be extended beyond 0; for while we may take 1 gram from 3, 2 or 1 gram, we cannot take it from 0. Hence the terms following zero exist only in the imagination; they may, therefore, be called imaginary. But instead of a series of weights, let us consider them as acting in a pan A of a balance containing weights in the other pan also; and for the purpose of illustration, let us suppose the distance passed over by the arms of the balance is proportional to the weight added or withdrawn, which indeed would be the case if a spring were adjusted to the axis. If the addition of

the weight n to the pan A moves the extremity of the arm A a distance n', the addition of the weights $2n$, $3n$, $4n$, ..., will cause this same extremity to move over the distances $2n'$, $3n'$, $4n'$, ..., which may be taken as measures of the weight in the pan A: this weight is zero when the pans are balanced. By placing the weights n, $2n$, $3n$ . . . ., in the pan A, we may obtain the results n', $2n'$ $3n'$ . . . . ., or, by starting with $3n'$ and withdrawing the weights, the results $2n'$, n', 0. But these results may be reached not only by taking weights out of the pan A, but also by adding them to the pan B. Now the addition of weights to the pan B can be continued indefinitely; and in so doing we shall obtain results expressed by $-n'$, $-2n'$, $-3n'$, ..., and these terms, called *negative*, will express quantities as real as did the positive ones. We, therefore, see that when two terms, numerically equal, have opposite signs, as $3n'$, $-3n'$, they designate the different positions of the balance arms, such that the extremity indicating the weight is in both cases equally distant from the point 0. This distance may be considered apart from *direction*, and be then called *absolute*.

Let us consider the origin of negative quantities in a case of another kind. If in counting a sum of money we adopt the franc piece as unity, we may operate successively by subtraction on this sum, and render it zero by taking away a certain number of francs. At this point the operation becomes impracticable, and, consequently,—1 franc,—2 francs, ... are imaginary quantities. Take now the nominal franc as unity, for the purpose of estimating a fortune made up of credit and debit. What we call a *diminution* of this fortune might take place either by a decrease in the number of francs on the credit side, or by an increase in the number on the debit side, and by continuing either process we should have a negative fortune of -100 francs, -200 francs, ... Such expressions signify that the num-

ber of francs of debt, considered abstractly, exceed by 100, 200, those of credit. Thus -100 francs, -200 francs, ..., which in the former case can express only imaginary quantities, here represent quantities as real as those denoted by positive expressions.

2. These ideas are very simple ; yet it is not so easy, as it at first seems, to set them forth clearly, and to give them the generality which their application requires. Moreover, the difficulty of the subject will not be questioned if we remember that the exact sciences had been cultivated for many centuries, and had made great progress before either a true conception of negative quantities was reached or a general method for their use had been devised. Moreover, it was not our intention to endeavor to state these principles more rigorously or more clearly than they are to be found in the works which deal with this subject ; but simply to make two remarks on negative quantities. First, that whether a nega-

tive quantity is real or imaginary,* depends upon the kind of magnitude measured; and, second, when we compare two quantities which are of a kind yielding negative values, the idea involved in their ratio is complex, including, 1° a relation dependent on number, considered *absolutely;* 2° a relation of *direction*, or of the sense in which they are estimated, a relation either of identity or opposition.

3. If now, setting aside the ratio of absolute magnitude, we consider the different possible relations of direction, we shall find them reducible to those expressed in the two following proportions:

$$+1 : +1 :: -1 : -1,$$
$$+1 : -1 :: -1 : +1.$$

* The sense in which these words are used is sufficiently determined by what precedes: the extension here given to their ordinary meaning seems permissible, and is moreover not wholly new. In optics, what is called the imaginary focus, as distinguished from the real, is the point of intersection of rays which have no existence, in a physical sense, and which can be considered, in some sort, as negative rays.

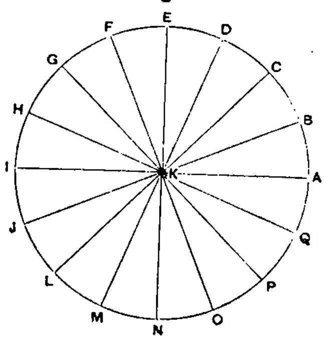
Taken directly and by inversion, these proportions show that the signs of the means are alike or different when those of the extremes are so. Now let it be required to find the geometrical mean between two quantities of different signs, that is, to find the value of x in the proportion

$$+1 : x :: x : -1.$$

Here we encounter a difficulty, as when we wished to continue the decreasing arithmetical progression beyond zero, for x cannot be made equal to any quantity, either positive or negative; but, as before, the quantity which was imaginary, when applied to certain magnitudes, became real when to the idea of *absolute number* we added that of *direction*, may it not be possible to treat this quantity, which is regarded imaginary, because we cannot assign it a place in the scale of positive and negative quantities, with the same success? On reflection this has seemed possible, provided we can devise a kind of quantity to which we may apply the idea of direction, so that having chosen two opposite directions, one for positive and one for negative values, there shall exist a third—such that the positive direction shall stand in the same relation to it that the latter does to the negative.

4. If now we assume a fixed point K (Fig. 1) and the line KA be taken as

Fig. 1.



positive unity, and we also regard its direction, from K to A, and write $\overline{KA}$ to distinguish it from the line KA as simply an absolute distance, negative

unity will be $\overline{KI}$, the vinculum having the same meaning as before, and the condition to be satisfied will be met by KE, perpendicular to the above and with a direction from K to E, expressed in like manner by $\overline{KE}$. For the direction of $\overline{KA}$ is to that of $\overline{KE}$ as is the latter to that of $\overline{KI}$. Moreover we see that this same condition is equally met by $\overline{KN}$, as well as by $\overline{KE}$, these two last quantities being related to each other as $+1$ and -1. They are, therefore, what is ordinarily expressed by $+\sqrt{-1}$, and $-\sqrt{-1}$. In an analogous manner we may insert other mean proportionals between the quantities just considered. Thus to construct the mean proportional between $\overline{KA}$ and KE, the line CKL must be drawn so as to bisect the angle AKE, and the required mean will be $\overline{KC}$ or KL. So the line GKP gives in like manner the means between $\overline{KE}$ and $\overline{KI}$, or between $\overline{KA}$ and $\overline{KN}$. We shall obtain in the same way $\overline{KB}$, $\overline{KD}$, $\overline{KF}$, $\overline{KH}$, $\overline{KJ}$, $\overline{KM}$, $\overline{KO}$, $\overline{KQ}$, as means between $\overline{KA}$ and KC, $\overline{KC}$ and KE . . . .

and so on. Similarly we might insert a greater number of mean proportionals between two given quantities, and the number of constructions involved in the solution would be equal to the number of ratios in the required series. Thus, for example, to construct two means, $\overline{KP}$, $\overline{KQ}$, between $\overline{KA}$ and $\overline{KB}$, we should have the three ratios $\overline{KA} : \overline{KP} :: \overline{KP} : \overline{KQ} :: \overline{KQ} : \overline{KB}$, and necessarily, angle AKP=angle $\overline{PKQ}$=angle $\overline{QKB}$, the vinculum indicating that these angles are similarly situated with respect to the bases AK, PK, QK. Now this may be effected in three ways, namely, by trisecting 1° the angle AKB; 2° the angle AKB increased by 360°; 3° the angle AKB increased by twice 360°, giving the three constructions of Fig. 2, 2 bis, 2 ter.*

* The principle on which these constructions rest, stated generally, is that the ratio of two radii $\overline{KP}$, $\overline{KQ}$, making an angle QKP, depends on this angle when these radii are considered as drawn in a certain direction, and that this ratio is the same as that of two other radii $\overline{KR}$, KS, making the same angle: but although this principle is, in a way, an extension of that on which the geometrical ratio of a positive and negative line was established, it is here only an hypothesis whose legitimacy must be proved, and whose consequences, till then, are to be independently confirmed.

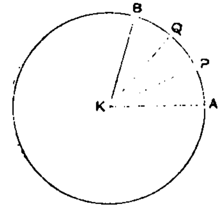


Fig. 2.

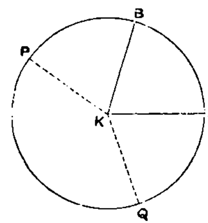


Fig. 2. bis.

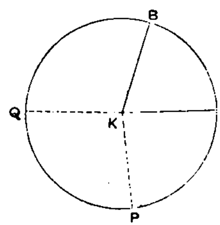


Fig. 2. ter.

5. Observe, further, that the relations just established between the quantities

$\overline{KA}$, $\overline{KB}$, $\overline{KC}$, ..., do not require that the directions which these quantities fundamentally involve should be estimated from a single point K; but that these relations are equally true for every such expression as $\overline{KA}$, indicating an absolute distance KA and taken in the same direction, as $\overline{K'A'}$, $\overline{K''A''}$, $\overline{K'''A'''}$, $\overline{BK}$, ... (Fig. 3). For,

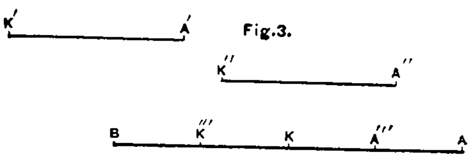


Fig. 3.

following with respect to this new quantity the same reasonings as before, we see that if $\overline{KA}$, $\overline{K'A'}$, $\overline{K''A''}$, ..., are each positive unity, $\overline{AK}$, $\overline{A'K'}$, $\overline{A''K''}$, are negative unities; that the mean proportional between $+1$ and -1 can be expressed by any line whatever, equal in length to the above and perpendicular to them in direction, and taken at pleasure in either of its two directions, and so on. To make this clear, consider a

particular case as, for example, a given force assumed as unity and represented by $\overline{KA}$, acting parallel to KA in the direction from K to A, its point of application being arbitrary; this unit force may be expressed by a line parallel to KA, with any point as an origin. The negative unit would be an equal force with a parallel action line, but acting from A towards K, and could likewise be represented by a line drawn from any point parallel to the former one, but in an opposite direction. All that is necessary, then, to the application of the principles already developed regarding radii is that the qualities, indicated by plus and minus, which we attribute to a certain quantity, should depend upon opposite directions between which there exists a mean; and that the relations between all lines which will represent such a quantity be then conceived as the same which existed between the radii.

6. From these reflections it follows that we may generalize the meaning of expressions of the form $\overline{AB}$, $\overline{CD}$, $\overline{KP}$,

. . . . . . , every such one representing a line of a certain length, parallel to a certain direction, the latter taken definitely in one of the two opposite senses which this direction presents, with any point as an origin; these lines themselves being capable of representing magnitudes of another kind. As they are to be the subject of the following investigations, it is proper to give them some special designation. They will be called *lines having direction*, or simply, *directed lines*.* They will be thus distinguished from *absolute* lines, whose length only is considered without regard to direction.†

7. Applying the terms of common

[* The *directed lines* of Argand are, of course, Hamilton's vectors, and the above principle is simply a statement of the fundamental conception of a vector, *i. e.* that all quantities having direction as well as magnitude are vectors, and that vectors are not changed by translation without rotation.—TRANS.]

† The expression *lines having direction* is only an abbreviation of *lines considered with reference to their direction*. This remark will show that we do not pretend to create a new nomenclature, but, by this denomination, both to avoid confusion and secure brevity.

usage to the different varieties of directed lines which arise in connection with a primitive unit $\overline{KA}$, it is seen, that every line parallel to the primitive direction is expressed by a real number, that those perpendicular to it are expressed by imaginaries of the form $\pm a\sqrt{-1}$, and, finally, that those having other directions are of the form $\pm a \pm b\sqrt{-1}$, and are composed of a real and imaginary part. But these lines are quantities quite as real as the positive unit; they are derived from it by the association of the idea of direction with that of magnitude, and are in this respect like the negative line, which has no imaginary signification. The terms *real* and *imaginary* do not therefore accord with the above exposition. It is needless to remark that the expressions *impossible* and *absurd*, sometimes met with, are still less appropriate. The use of these terms in the exact sciences in any other sense than that of *not true* is perhaps surprising.*

* There was a time, when led by the very nature of the case to admit negative values in the discussion of

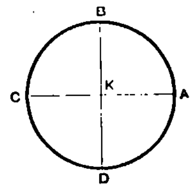
An absurd quantity would be one whose existence involved the truth of a false proposition; as, for example, the quantity x, satisfying at once $x=2, x=3$, whence $2=3$. The admission of such a quantity into the calculus would entail consequences as contradictory as $2=3$; but the results obtained from the use of the so-called imaginaries are in all respects conformable to those derived from reasonings in which only real quantities appear. We might thus foresee the impropriety of a nomenclature which classifies truly absurd quantities and the even roots of negative quantities together, and it was a consciousness of this impropriety which first gave rise to the ideas developed in this essay. It is thus that we are led to a new nomenclature.†

abstract quantities, geometers, having apparently some difficulty in imagining that *less than nothing* could be *anything*, applied to such values the term *fausses*. The use of this word, in its original vicious sense, ceased when the conception under which it arose was rectified.

† It is almost needless to observe that we refer only to the confusion which arises from the terms, and that a corresponding confusion of ideas is not implied.

It is to be observed that while there exists an infinite variety of directed lines, practically they are all referred, as will be shortly shown, to $\overline{KA}$, $\overline{KC}$, $\overline{KB}$, $\overline{KD}$, the position unit being $\overline{KA}$ the negative $\overline{KC}$ and the means $\overline{KB}$ and $\overline{KD}$ (Fig. 4).

Fig. 4.



It is, furthermore, convenient to classify any two opposite directions under one head, to which we shall apply the term *order*. The primitive $\overline{KA}$ with its negative $\overline{KC}$ we shall designate as the *prime order*, and the means $\overline{KB}$ and $\overline{KD}$ as the *medial order*. We shall speak of a *prime quantity* or *medial quantity* when we refer to one of a *prime* or *medial order*, respectively. These terms are derived from the mode of generation of these quantities, and from the conception under which they are regarded real. We might apply the general term *intermedials* to all others which it is not necessary to designate specially.*

8. In accordance with what precedes, we may also modify the language of so-called *imaginaries* in such a way as to render this part of the subject more simple. In writing $+a\sqrt{-1}$ or $-a\sqrt{-1}$, we indicate explicitly the way in which the quantity is generated, which in certain cases may be useful: but ordinarily we leave the mode of generation out of consideration, and $\sqrt{-1}$ is only a par-

*It has been already remarked that the relations said to exist between lines, when we take their directions into account, cannot as yet be regarded other than hypothetical. It is, therefore, very far from our purpose to propose the substitution of the nomenclature above described for that commonly employed; but to make use of it only because, in general, it is desirable to avoid the employment of terms whose real meaning is at variance with the ideas we wish to express, even when we are concerned with an hypothesis.

ticular kind of unit to which the number a is referred. It is, therefore, not absolutely essential to keep the mode of generation in view. Again, the expression $a\sqrt{-1}$ shows $\sqrt{-1}$ to be a multiplier of a; but really $\sqrt{-1}$, in $a\sqrt{-1}$, is no more a factor than is $+1$ in $+a$, or -1 in $-a$. Now we do not write $+1.a, -1.a$, but simply $+a, -a$, and the sign which precedes a itself indicates what kind of a unit this number expresses. We may then apply a similar method to imaginary quantities, writing for example $\sim a$ and $\dashv\!\vdash a$ instead of $+a\sqrt{-1}$ and $-a\sqrt{-1}$, the signs $\sim$ and $\dashv\!\vdash$ being reciprocally positive and negative. To multiply these signs, we observe that either multiplied by itself gives $-$, and, consequently, multiplied by each other they give $+$. Moreover, a single rule, applicable to any number of factors, may be established; let every straight line, horizontal or vertical, in the signs to be multiplied, have a value 2, and every curved one a value 1; we shall have for the four signs the following values:

$$\sim\,=1,\quad -\,=2,\quad \intercal=3,\quad +\,=4.$$

Then take the sum of the values of all the factors and subtract as many times 4 as is necessary to make the remainder one of the numbers 1, 2, 3, 4; this remainder will be the value of the sign of the product; and so, for division, subtract the sum of the sign values of the divisor from that of the dividend, having added if necessary a multiple of 4 to the latter, and the remainder will indicate the sign of the quotient. It is to be noticed that these operations are those of multiplication and division by logarithms; this analogy will be brought more fully into view.

These new signs would abridge the notation,* and perhaps render the calculus of imaginaries more convenient, errors of sign being sometimes easily

* The quantity $m \pm n\sqrt{-1}$ being denoted by $m\sim n$, or by $m \intercal n$, the single sign $\sim$, or $\intercal$, replacing the four signs $+$, $\sqrt{}$, $-$, 1.

made.* We shall employ them in what follows, without implying on that account that they should be adopted. Doubtless to every innovation, even a rational one, there is an intrinsic objection; but no progress would be made if they were rejected, for the only reason that they are contrary to usage, and their trial, at least, is permissible.

9. We are now to examine the various ways in which directed lines are combined by addition and multiplication, and to determine the resulting constructions. Suppose, first, that we have to add to the positive prime line $\overline{KP}$ (Fig. 5) the line $\overline{KQ}$, also a positive

Fig. 5.

* For example, let it be required to multiply $-m\sqrt{-c}$ by $+n\sqrt{-cd}$. The product of the two coefficients is $-mn$; that of the two radicals is $-c\sqrt{d}$; and the final product is $+mnc\sqrt{d}$. In the new notation the two factors are $\sim m\sqrt{c}$, $\intercal n\sqrt{cd}$, or $\intercal m\sqrt{c}$, $\sim n\sqrt{cd}$, and by the rule we at once obtain $\intercal mnc\sqrt{d}$. This advantage—if it be one—would not exist for an experienced calculator, who by a simple inspection of the factors would read the product; but not every one possesses this faculty.

prime; the construction would not differ from that of finding the sum of the absolute lines KP, KQ; it consists in laying off the distance PR=KQ on the prolongation of KP. We then have $\overline{KP}+\overline{KQ}=\overline{KP}+\overline{PR}=\overline{KR}$. To add a negative prime line $\overline{QK}$ to another $\overline{PK}$, the construction is the same, but in the opposite direction, and we should have $\overline{PK}+\overline{QK}=\overline{PK}+\overline{RP}=\overline{RK}$. In general, if we are to add two lines of the same direction, $\overline{AB}$, $\overline{AC}$, we take in this direction, PQ = AB, QR = AC, and we have $\overline{PQ}+\overline{QR}=\overline{AB}+\overline{AC}=\overline{PR}$. If we are to add to the positive line $\overline{KP}$ the negative $\overline{QK}$, we take a distance PS=QK in the negative direction from P, and obtain $\overline{KP}+\overline{QK}=\overline{KS}=\overline{QP}$. The same course is pursued for any other order.

Now, the principle underlying these constructions is that we regard P, the final point of $\overline{KP}$, as the initial point of the line to be added, and that we take respectively for the initial and final points of the sum, the initial point of

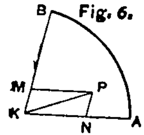
$\overline{KP}$ and the final point of the added line. Applying this same principle to lines of other orders, we conclude that K, P, R, being any points whatever, we always have $\overline{KP}+\overline{PR}=\overline{KR}$; and as each of the lines $\overline{KP}$, $\overline{PR}$ may also be the sum of two lines, as $\overline{KM}+\overline{MP}$, $\overline{PN}+\overline{NR}$, M and N being arbitrarily chosen, we conclude that, in general, A, B; M, N, O, ..., R, S, T being any points whatever, $\overline{AB}=\overline{AM}+\overline{MN}+\overline{NO}+\overline{O\;.\;.\;.}+\overline{.\;.\;.}+\overline{.\;.\;.\;R}+\overline{RS}+\overline{ST}+\overline{TB}$. The points A, B, M, ... may coincide or be so situated that the lines $\overline{AM}$, $\overline{MN}$, ... coincide, intersect, etc. These circumstances are matters of indifference.*

10. Every directed line may thus be decomposed in an infinite number of ways. To decompose, for example, the line $\overline{KP}$ (Fig. 6) into two, one of an order $\overline{KA}$, the other of an order $\overline{KB}$; draw, through P, PN parallel to BK, and we have $\overline{KP}=\overline{KN}+\overline{NP}$. Or we might

* This rule is reached by induction, and what was said in the note to No. 4, on the geometrical ratio of directed lines, is here applicable.

draw PM parallel to KA, and then $\overline{KP}=\overline{KM}+\overline{MP}$; but these two expressions are identical, because $\overline{KM}=\overline{NP}$ and $\overline{KN}=\overline{MP}$. As there is no other way to

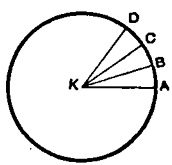
Fig. 6.



effect the proposed decomposition, we conclude that, if A and A′ are of the order a, B and B′ of another order b, and we have the equation $A+B=A'+B'$, then $A=A'$, $B=B'$.

11. Let us now pass to the multiplication of directed lines, and let us first construct the product $\overline{KB}\times\overline{KC}$ (Fig. 7),

Fig. 7.



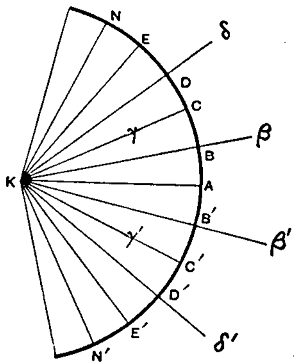
the factors being units, but not prime units. Construct the angle $\overline{CKD}=\overline{AKB}$.

From what was said in No. 4, Note I, we have $\overline{KA}:\overline{KB}::\overline{KC}:\overline{KD}$, whence $\overline{KA}\times\overline{KD}=\overline{KB}\times\overline{KC}$; but $\overline{KA}=+1$, hence $\overline{KB}\times\overline{KC}=\overline{KD}$. Therefore, to construct the product of two directed radii, lay off, from the origin of arcs, the sum of the arcs corresponding to each radius, and the extremity of the arc thus laid off will determine the position of the radius of the product; this, as before, is logarithmic multiplication. It is unnecessary to show that this rule applies to any number of factors. If the factors are not units, they can be put under the form $m.\overline{KB}$, $n.\overline{KC}$, . . ., m and n being coefficients or positive prime lines, and the product would be $(mn\ldots).(\overline{KB}.\overline{KC}\ldots)=(mn\ldots).\overline{KP}$. Now, the product of the positive prime line $(mn\ldots)$ by the radius $\overline{KP}$ is this very line, drawn in the direction of this radius. Division is the inverse of this operation, and its explanation in detail is unnecessary.

12. By means of these rules we may operate on directed lines as on absolute

ones. We now proceed to some applications of the principles already laid down, and we shall first state some immediate consequences which are of most frequent use.

Fig. 8.



§1. If AB, BC, ... , EN (Fig. 8) are equal arcs, n in number, and we make $\overline{KB}=u$, we shall have $\overline{KC}=u^2$, $\overline{KD}=u^3$, ... , $\overline{KN}=u^n$.

§2. If we lay off the arcs below KA, as AB′, B′C′, ... , E′N′, we shall have

$$\overline{KB'}=\frac{1}{u},\ \overline{KC'}=\frac{1}{u^2},\ \ldots,\ \overline{KN'}=\frac{1}{u^n}.$$

§3. Hence

$$\frac{\overline{KB}}{\overline{KB'}}=u^2,\ \frac{\overline{KC}}{\overline{KC'}}=u^4,\ \ldots,\ \frac{\overline{KN}}{\overline{KN'}}=u^{2n}.$$

§4. If, on corresponding radii, we take $K\beta=K\beta'$, $K\gamma=K\gamma'$, $K\delta=K\delta'$, ... , the distances $K\beta$, $K\gamma$, $K\delta$, . . . being arbitrary, we obtain

$$\frac{\overline{K\beta}}{\overline{K\beta'}}=u^2,\ \frac{\overline{K\gamma}}{\overline{K\gamma'}}=u^4,\ \frac{\overline{K\delta}}{\overline{K\delta'}}=u^6,\ \ldots\ldots$$

§5. If on the radii $\overline{KA}$, $\overline{KM}$, $\overline{KN}$ as bases, similar and equal figures be constructed, $\overline{a}$, $\overline{m}$ and $\overline{n}$ being homologous lines, then $\overline{m}=\overline{a}\times\overline{KM}$, $\overline{n}=\overline{a}\times\overline{KN}$,

whence $\dfrac{\overline{m}}{\overline{KM}}=\dfrac{\overline{n}}{\overline{KN}}$, or $\overline{m}.\overline{KN}=\overline{n}.\overline{KM}$.

§6. MN being any arc of the circumference, it may at times be convenient to denote, in general, by $\overline{K.MN}$ the directed radius drawn though the extremity B of the arc AB=MN, A always being the origin of arcs. We should thus have

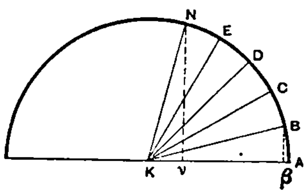
$$\overline{K.MN}\times\overline{K.PQ}=\overline{K.(MN+PQ)},$$

and
$$\frac{\overline{K.MN}}{\overline{K.PQ}}=\overline{K(MN-PQ)}.$$

§7. If $\overline{KB}$ has the same direction as $\overline{PQ}$, we have $\overline{PQ}=PQ\times\overline{KB}$; for the absolute line PQ may be regarded as positive prime.

§8. If we have the equation $r'.\overline{PQ}=r''.\overline{MN}$; r', r'' being unknown directed radii, and $\overline{PQ}$, $\overline{MN}$ lines of the same

Fig. 9.



direction, or absolute lines, it follows that $r'=r''$, and consequently $\overline{PQ}=\overline{MN}$, or PQ=MN.

13. Now let AB, BC, ... , EN (Fig. 9) be equal arcs, n in number;

then $\overline{KN}=\overline{KB}^n$; but $\overline{KN}=\overline{K\nu}+\overline{\nu N}$, and $\overline{KB}=\overline{K\beta}+\overline{\beta B}$; hence

$$\overline{K\nu}+\overline{\nu N}=(\overline{K\beta}+\overline{\beta B})^n.$$

Let the arc AB$=a$, and, therefore, AN$=na$; then $\overline{K\beta}=\cos a$, $\overline{K\nu}=\cos na$, $\overline{\beta B}=\sim\sin a$, $\overline{\nu N}=\sim\sin na$; and the above equation becomes

$$\cos na\sim\sin na=(\cos a\sim\sin a)^n.$$

This theorem, expressed in the ordinary notation by

$$\cos na\pm\sqrt{-1}\sin na=(\cos a\pm\sqrt{-1}\sin a)^n,$$

is a fundamental one in the theory of circular functions; among its uses is the expansion of $\sin x$ and $\cos x$ into series.

Developing the binomial, equating separately the terms of the same order, and dividing by ~1 the equation between the medials, we have the expressions for $\cos na$ and $\sin na$; then making $na=x$, and supposing n to increase and a to diminish, x remaining constant, we have, at the limit,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \cdots,$$

$$\sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \cdots.$$

14. From $\overline{KN} = \overline{KB}^n$ we have $\overline{KB} = \overline{KN}^{\frac{1}{n}}$, whence

$$\overline{K\beta} + \overline{\beta B} = (\overline{K\nu} + \overline{\nu N})^n = \overline{K\nu}^{\frac{1}{n}}$$

$$+ \frac{1}{n}\overline{K\nu}^{\frac{1}{n}-1} . \overline{\nu N} + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2}\overline{K\nu}^{\frac{1}{n}-2}\overline{\nu N}^2$$

$$+ \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{2.3}\overline{K\nu}^{\frac{1}{n}-3} . \overline{\nu N}^3 + \cdots$$

$$= \overline{K\nu}^{\frac{1}{n}}\left\{1 + \frac{1}{n}\frac{\overline{\nu N}}{\overline{K\nu}} + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2}\cdot\left(\frac{\overline{\nu N}}{\overline{K\nu}}\right)^2 + \right.$$

$$\left. \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{2.3}\cdot\left(\frac{\overline{\nu N}}{\overline{K\nu}}\right)^3 + \cdots\right\}.$$

Substituting the preceding values of $\overline{K\nu}$ and $\overline{\nu N}$, and noticing that

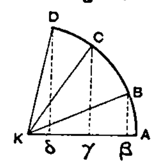
$$\frac{\overline{\nu N}}{\overline{K\nu}} = \frac{\sim\sin na}{\cos na} = \sim\tan na,$$

equating separately the terms of the same order, multiplying the equation between the medials by $\frac{n}{\sim 1} = ⟂ n$, and making the same supposition as before, there results

$$x = \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \frac{\tan^7 x}{7} + \cdots.$$

15. Let (Fig. 10) the arcs $AB = a$,

Fig. 10.



$AC = b$, and let CD be taken equal to AB. Then (No. 11), $\overline{KD} = \overline{KB} \times \overline{KC}$. But

$$\overline{KD} = \overline{K\delta} + \overline{\delta D} = \cos(a+b) \sim \sin(a+b),$$
$$\overline{KB} = \overline{K\beta} + \overline{\beta B} = \cos a \sim \sin a,$$
$$\overline{KC} = \overline{K\gamma} + \overline{\gamma C} = \cos b \sim \sin b; \text{ hence}$$

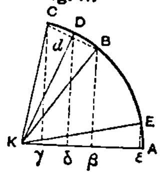
$$\cos(a+b) \sim \sin(a+b) = (\cos a \sim \sin a)(\cos b \sim \sin b).$$

Expanding the second member and equating the orders separately, we have

$$\cos(a+b)=\cos a\cos b-\sin a\sin b,$$
$$\sin(a+b)=\cos a\sin b+\sin a\cos b.$$

16. Let AC $=a$, AB $=b$ (Fig. 11); draw

Fig. 11.



the chord BC, and the radius KD, bisecting the angle BKC. Make AE $=$ BD $=\dfrac{a-b}{2}$, and draw KE and Eε. Then $\overline{K\gamma}+\overline{\gamma C}-(\overline{K\beta}+\overline{\beta B})=\cos a\sim\sin a-(\cos b\sim\sin b)=\cos a-\cos b\sim(\sin a-\sin b)=\overline{KC}-\overline{KB}=\overline{KC}+\overline{BK}=\overline{BC}=\overline{2dC}$ $=$[No. 12, § 5] $2\,\overline{\varepsilon E}\times\overline{KD}=$

$$\sim 2\sin\frac{a-b}{2}\left(\cos\frac{a+b}{2}\sim\sin\frac{a+b}{2}\right);$$

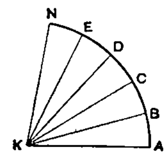
whence

$$\cos a-\cos b=-2\sin\frac{a-b}{2}.\,\sin\frac{a+b}{2},$$

$$\sin a-\sin b=+2\sin\frac{a-b}{2}.\,\cos\frac{a+b}{2}.$$

17. Divide the arc AN (Fig. 12) into n equal parts. The radii $\overline{KA}$, $\overline{KB}$, $\overline{KC}$, . . . , $\overline{KN}$ are in geometrical progression, but the corresponding arcs are in arithmetical progression, and may therefore

Fig. 12.



be taken for the logarithms of these radii. Put mAN $=\log\overline{KN}$, m being the arbitrary modulus; we then have $\log\overline{KN}=m.$AN$=mn$AB. Making n infinity, so that the arc AB may be regarded as a right line perpendicular to KA, we have $\overline{AB}=\sim$AB, or $\overline{AB}=\top$AB, and $\log\overline{KN}=\top mn.$AB, or $\log\overline{KN}=mn.\overline{AB}$; for since m is arbitrary, we may substitute m in place of $\top m$. Now $\overline{AB}=\overline{AK}+\overline{KB}$ $=-1+\overline{KN}^{\frac{1}{n}}$; hence

$\log \overline{KN} = mn\left(-1+\overline{KN}^{\frac{1}{n}}\right)$, and, putting $KN = 1 + x$, $\log(1+x) = mn$

$$\left\{-1+1+\frac{1}{n}x+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2}x^2+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{2\,.\,3}x^3+\ldots\right\}$$

$$= m\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right).$$

18. Let us now divide the two equal arcs AN, AN′ (Fig. 13) into n equal parts; draw the tangent nn' and the secants Kb, Kc, . . ., Kn; Kb', Kc', ..., Kn'.

We have seen (No. 12, § 4) that, when $\overline{KB} = u$,

$$\frac{\overline{Kb}}{\overline{Kb'}} = u^2, \frac{\overline{Kc}}{\overline{Kc'}} = u^4, \ldots, \frac{\overline{Kn}}{\overline{Kn'}} = u^{2n}.$$

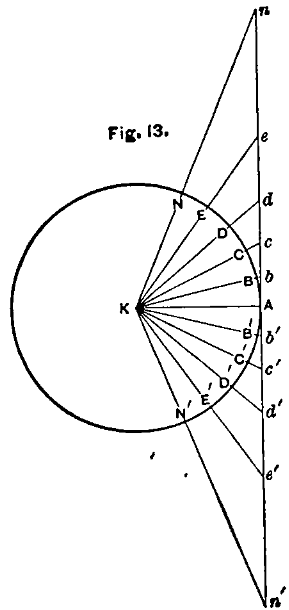
Thus, as before, the quantities

$$\frac{\overline{KA}}{\overline{KA'}}, \frac{\overline{Kb}}{\overline{Kb'}}, \frac{\overline{Kc}}{\overline{Kc'}}, \ldots, \frac{\overline{Kn}}{\overline{Kn'}}$$

are in geometrical progression, and the

corresponding arcs may be taken for their logarithms, as for example

Fig. 13.



$$AN = m \log \frac{Kn}{Kn'}.$$

Let AN $= x$, and consequently

$$\overline{\mathrm{K}n}=\overline{\mathrm{KA}}+\overline{\mathrm{A}n}=1\sim\tan x,$$
$$\overline{\mathrm{K}n'}=\overline{\mathrm{KA}}+\overline{\mathrm{A}n'}=1\dotplus\tan x;$$

then we have at once

$$x=m\log\frac{1\sim\tan x}{1\dotplus\tan x}.$$

But we have seen the arc

$$x=\tan x-\frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}-\quad.\;.\;.\;.\;.\;;$$

hence

$$m\log\frac{1\sim\tan x}{1\dotplus\tan x}=\tan x-\frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}-\quad.\;.\;.\;.\;.$$

Putting $\sim\tan x=z$, this becomes

$$m\log\left(\frac{1+z}{1-z}\right)=\sim\left(z+\frac{z^3}{3}+\frac{z^5}{5}+\frac{z^7}{7}+\ldots\right),$$

or, dividing both members by ~ 1, and noticing that, m being arbitrary, m may be substituted for $\dfrac{m}{\sim 1}$,

$$m\log\left(\frac{1+z}{1-z}\right)=z+\frac{z^3}{3}+\frac{z^5}{5}+\frac{z^7}{7}+\quad.\;.\;.\;.$$

19. Resuming the equation

$$x=m\log\frac{1\sim\tan x}{1\dotplus\tan x},$$

and making $\sim\tan x=z$ as before, let us substitute $\dfrac{m}{\sim 2}$ for m, or, what is the same thing, make $x=\sim 2x$ in the first member. These changes give

$$\sim 2x=m\log\frac{1+z}{1-z}=\log(1+2z+2z^2+2z^3+\ldots),$$

and

$$\sim 2nx=m\log(1+2z+2z^2+2z^3+\ldots)^n.$$

Now make $\sim 2nx=y$, and suppose n to increase and x to decrease without limit, y remaining constant; $z=\sim\tan x$ will then be infinitely small; hence, in the second member of the development the terms following $2z$ may be omitted, under which supposition the equation reduces to $y=m\log(1+2z)^n$.

The same supposition gives $z=\sim\tan x=\sim x$, $2nz=\sim 2nx=y$, and, therefore, $2z=\dfrac{y}{n}$. We may then write

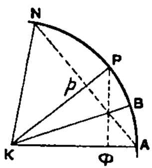
$$y=m\log\left(1+\frac{y}{n}\right)^n$$
$$=m\log\left(1+n.\frac{y}{n}+\frac{n(n-1)}{1.2}\cdot\frac{y^2}{n^2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{y^3}{n^3}+\ldots\right),$$

and, finally, since $n=\infty$,

$$y=m\log\left(1+y+\frac{y^2}{2}+\frac{y^3}{1.2.3}+\frac{y^4}{1.2.3.4}\cdots\right).$$

20. Suppose the arc AN (Fig. 14) to

Fig. 14.



be divided into an infinite number n of equal parts, of which AB is the first. Take $AP=\frac{1}{2}AN$, and draw AN, KP and Pφ. We have

$$\overline{AB}=\overline{AK}+\overline{KB}=-1+\overline{KN}^{\frac{1}{n}}$$

$$=-1+(\overline{KA}+\overline{AN})^{\frac{1}{n}}=-1+(1+\overline{AN})^{\frac{1}{n}}.$$

$$=-1+1+\frac{1}{n}.\overline{AN}+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2}.\overline{AN}^2$$

$$+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{2.3}.\overline{AN}^3+\quad.\quad.\quad.$$

$$=\frac{1}{n}\left(\overline{AN}-\frac{\overline{AN}^2}{2}+\frac{\overline{AN}^3}{3}-\frac{\overline{AN}^4}{4}+\ .\ .\ .\ \right).$$

and $\quad n\,.\,\overline{AB}=\overline{AN}-\dfrac{\overline{AN}^2}{2}+\dfrac{\overline{AN}^3}{3}-\dfrac{\overline{AN}^4}{4}+\ .\ .\ .\ .\ ;$

whence $n\,.\,\overline{AB}=\sim n\,.\,AB=\sim$arc AN, and $\overline{AN}=2\overline{pN}=$[No. 12, §5] $2\overline{\varphi P}\times\overline{KP}$

$$=\sim 2\sin\frac{AN}{2}.\left(\cos\frac{AN}{2}\sim\sin\frac{AN}{2}\right),$$

or, putting the arc $\dfrac{AN}{2}=a$, $\overline{AN}=\sim 2\sin a$ $(\cos a\sim\sin a)$, and therefore

$$\overline{AN}^2=-(2\sin a)^2(\cos 2a\sim\sin 2a),$$
$$\overline{AN}^3=+(2\sin a)^3(\cos 3a\sim\sin 3a),$$
$$\overline{AN}^4=+(2\sin a)^4(\cos 4a\sim\sin 4a),$$
$$\overline{AN}^5=\sim\ .\ .\ .\ .\ .\ .\ .\ .$$

Substituting these values in the above series, retaining the medial terms only, since this series is equal to $n.\overline{AB}=\sim 2a$, and dividing by ~ 1, we have

$$2a=2\sin a.\cos a+\frac{(2\sin a)^2\sin 2a}{2}-\frac{(2\sin a)^3\cos 3a}{3}-\frac{(2\sin a)^4\sin 4a}{4}+\ .\ .\ .\ .$$

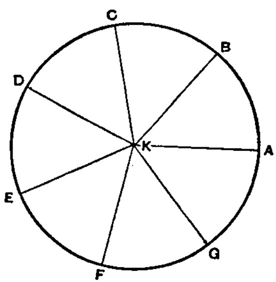
Since the sum of the prime terms must be zero, we have

$$0=-2\sin a.\sin a+\frac{(2\sin a)^2\cos 2a}{2}+\frac{(2\sin a)^3\sin 3a}{3}-\frac{(2\sin a)^4\cos 4a}{4}-\ldots,$$

an equation which may be divided by $2\sin a$.

21. Divide the circumference (Fig. 15)

Fig. 15.



into n equal parts, AB, BC, . . . GA; n is now a finite quantity. We propose to find the sum S of the mth powers of the radii $\overline{\mathrm{KA}}, \overline{\mathrm{KB}}, \ldots \overline{\mathrm{KG}}$.

Let $\overline{\mathrm{KB}}=u$, whence $\overline{\mathrm{KC}}=u^2$, $\overline{\mathrm{KD}}=u^3, \ldots, \overline{\mathrm{KG}}=u^{n-1}$, $\overline{\mathrm{KA}}=u^n=1$; then

$$\mathrm{S}=1+u^m+u^{2m}+\ \ldots\ +u^{(n-1)m},$$

and

$$u^m\mathrm{S}=u^m+u^{2m}+\ldots+u^{(n-1)m}+u^{nm};$$

but $\qquad u^{nm}=(u^n)^m=1^m=1;$

hence $u^m\mathrm{S}=\mathrm{S}$, and $(u^m-1)\mathrm{S}=o$.

If $u^m=1$, this is an identical equation, without meaning; but, in this case, $u^{2m}=1$, $u^{3m}=1, \ldots$; hence $\mathrm{S}=n$. In all other cases $\mathrm{S}=o$.

If we denote by $\mathrm{P}', \mathrm{P}'', \mathrm{P}''', \ldots, \mathrm{P}^{(n)}$ the sum of the first, second, ... , nth powers of given quantities, and by $\Pi', \Pi'', \Pi''', \ldots, \Pi^{(n)}$ the sum of the products of the same quantities taken one and one, two and two, ... , n and n, it is well known that

$$n\Pi^{(n)}=\mathrm{P}'\Pi^{(n-1)}-\mathrm{P}''\Pi^{(n-2)}+\mathrm{P}'''\Pi^{(n-3)}-\ \ldots\ \pm\mathrm{P}^{(n-3)}\Pi'''\mp\mathrm{P}^{(n-2)}\Pi''\pm\mathrm{P}^{(n-1)}\Pi'\mp\mathrm{P}^{(n)},$$

the upper and lower signs corresponding to the cases in which n is even or odd, respectively. The demonstration of this theorem may be reduced to a simple algebraic transformation. If we apply it to the radii $\overline{KA}, \overline{KB}, \ldots, \overline{KG}$, which are n in number, we shall obtain $P'=o$, $P''=o$, $P'''=o$, $\ldots$, $P^{(n-1)}=o$ $P^{(n)}=n$; whence

$$\Pi'=o,\ \Pi''=o,\ \ldots,$$
$$\Pi^{(n-1)}=o,\ n\Pi^{(n)}=\mp n,$$
$$\text{and } \Pi^{(n)}=\mp 1=-(-1)^n.$$

These properties may also be derived from the equation $x^n-1=o$, whose roots are $\overline{KA}, \overline{KB}, \ldots, \overline{KG}$.

22. Let us now assume some point other than the center K (Fig. 16), as V, and find the product of VA, VB, VC, ... VG.

Since $\overline{VA}=\overline{VK}+\overline{KA}$, and $\overline{VB}=\overline{VK}+\overline{KB}$, ... we have

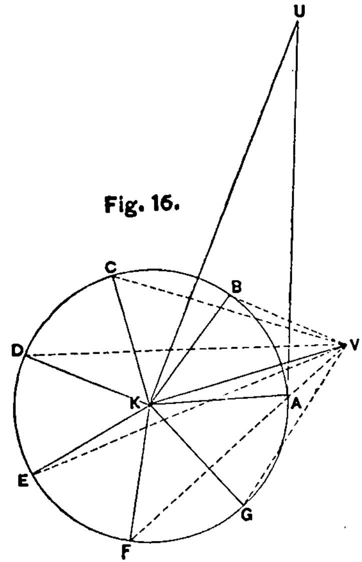
$$\overline{VA}.\overline{VB}.\overline{VC} \ \ldots \ \overline{VG}=$$
$$(\overline{VK}+\overline{KA})(\overline{VK}+\overline{KB}) \ldots$$
$$(\overline{VK}+\overline{KG})=\overline{VK}^n+\Pi'.\overline{VK}^{n-1}+$$
$$\Pi''.\overline{VK}^{n-2}+\ldots \Pi^{(n-1)}.\overline{VK}+\Pi^{(n)}.$$

Now we have just seen that the coefficients Π', Π'', Π''', . . . , to $\Pi^{(n-1)}$ are zero, and that $\Pi^{(n)}=-(-1)^n$. Hence

$$\overline{VA}.\overline{VB}.\overline{VC} \ldots \overline{VG}=\overline{VK}^n-(-1)^n$$
$$=(-\overline{KV})^n-(-1)^n=(\overline{KV}^n-1)(-1)^n,$$

To construct $\overline{KV}^n$, lay off $\overline{AKU}$ equal to n times $\overline{AKV}$. Make $KU=KV^n$, whence $\overline{KU}^n=\overline{KV}^n$. Therefore

Fig. 16.



$$\overline{VK}^n-1=\overline{KU}-1=\overline{KU}-\overline{KA}$$
$$=\overline{KU}+\overline{AK}=\overline{AU},$$

and $\overline{VA}.\overline{VB}.\overline{VC} \ldots \overline{VG}=(-1)^n\overline{AU}$.

If we consider VA, VB, ... , VG and

$\overline{\mathrm{AU}}$ as positive prime lines, we may make $\overline{\mathrm{VA}}=r'.\mathrm{VA}, \overline{\mathrm{VB}}=r''.\mathrm{VB}, \ldots, \overline{\mathrm{VG}}=r^{(n)}\mathrm{VG}$ and $\overline{\mathrm{AU}}=\rho.\mathrm{AU}$; $\rho, r', r'', \ldots, r^{(n)}$ being directed radii or the roots of unity. We should then have

$$r'.r''.r'''\ldots r^{(n)}.\mathrm{VA.VB.VC}\ldots\mathrm{VG} = (-1)^n\rho.\mathrm{AU},$$

and (No. 12, § 8), VA.VB .... VG=AU. For example, let $\mathrm{KV}=x$, $\mathrm{KU}=x^n$, the angle $\mathrm{AKV}=a$, the angle $\mathrm{AKU}=na$, the angle $\mathrm{AKB}=\dfrac{2\pi}{n}$. We should then find

$$\mathrm{AU}^2=x^{2n}-2x^n\cos na+1,$$

$$\mathrm{VB}^2=x^2-2x\cos\left(a-\frac{2\pi}{n}\right)+1,$$

$$\mathrm{VC}^2=x^2-2x\cos\left(a-\frac{4\pi}{n}\right)+1,$$

$$\mathrm{VD}^2=x^2-2x\cos\left(a-\frac{6\pi}{n}\right)+1,$$

$$\ldots\ldots\ldots\ldots,$$

$$\mathrm{VA}^2=x^2-2x\cos\left(a-\frac{2\pi n}{n}\right)+1 = x^2-2x\cos a+1,$$

and, squaring the equation VA.VB ... VG=AU, we shall have

$$x^{2n}-2x^n\cos na+1= \left\{x^2-2x\cos\left(a-\frac{2\pi}{n}\right)+1\right\} \times\left\{x^2-2x\cos\left(a-\frac{4\pi}{n}\right)+1\right\} \times\left\{x^2-2x\cos\left(a-\frac{6\pi}{n}\right)+1\right\} \times\ldots\ldots\times(x^2-2x\cos a+1),$$

there being n factors in the second member.

The development of the rational factors of the first or second degree of the binomials x^n+1, x^n-1, is obtained by making $\cos na=1$ and $\cos na=0$ in this formula. On this well known fact it is unnecessary to be more specific.

23. If V (Fig. 17) be on the circumference, we have

$$\mathrm{AU}=2\sin\frac{na}{2},$$

$$\mathrm{VA}=2\sin\frac{a}{2},$$

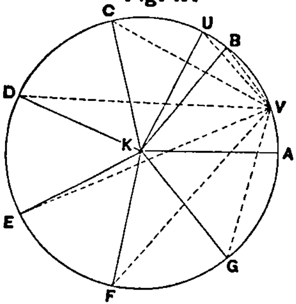
$$\mathrm{VB}=2\sin\left(\frac{\pi}{n}-\frac{a}{2}\right),$$

$$VC = 2\sin\left(\frac{2\pi}{n} - \frac{a}{2}\right),$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ,$$

$$VG = 2\sin\left\{\frac{(n-1)\pi}{n} - \frac{a}{2}\right\}.$$

Fig. 17.



Hence, substituting a for $\frac{a}{2}$ and

$$\sin\left(\frac{n\pi}{n} - a\right) = \sin(\pi - a)$$

for $\sin a$, there results

$2\sin na =$

$$2^n.\sin\left(\frac{\pi}{n} - a\right)\sin\left(\frac{2\pi}{n} - a\right)\sin\left(\frac{3\pi}{n} - a\right)$$

$$\cdot\ \cdot\ \times\sin\left\{\frac{(n-1)\pi}{n} - a\right\}\sin\left(\frac{n\pi}{n} - a.\right)$$

Making $a = \frac{\pi}{2n} - b$, we obtain $na = \frac{\pi}{2} - nb$ and $\sin na = \cos nb$. The substitution of these values gives

$2\cos nb = 2^n \cos$

$$\left\{\frac{(n-1)\pi}{2n} - b\right\}\cos\left\{\frac{(n-3)\pi}{2n} - b\right\}$$

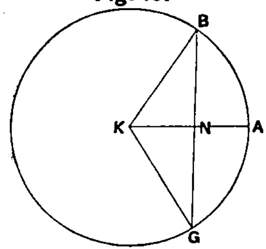
$$\times\cos\left\{\frac{(n-5)\pi}{2n} - b\right\}\ldots\times\cos$$

$$\left\{\frac{[n-(2n-3)]\pi}{2n} - b\right\}\times\cos$$

$$\left\{\frac{[n-(2n-1)]\pi}{2n} - b\right\}.$$

24. In Fig. 18, make the arc AB = arc

Fig. 18.



AG; then $\overline{KN} = \overline{KB} + \overline{BN}$ and $\overline{KN} = \overline{KG}$

$+\overline{GN}$; whence we obtain, observing that $\overline{BN}+\overline{GN}=0$, and that $\overline{KG}=\overline{KB}^{-1}$,

$$2\,\overline{KN}=\overline{KB}+\overline{KB}^{-1}.$$

Raising both members of this equation to the nth power, n being a whole number, it becomes

$$(2\overline{KN})^n=\overline{KB}^n+n\overline{KB}^{n-2}+\frac{n(n-1)}{2}.\overline{KB}^{n-4}+\ldots+\frac{n(n-1)}{2}.\overline{KB}^{-n+4}+n\overline{KB}^{-n+2}+\overline{KB}^{-n}.$$

Making the arc $AB=a$, then $\overline{KN}=\cos a$, $\overline{KB}=\cos a\sim\sin a$, and, in general, $\overline{KB}^m=\cos ma\sim\sin ma$. Substituting these values in the above equation, and, since the first member contains no medial terms, suppressing those of the second member, we have

$$(2\cos a)^n=\cos na+n\cos(n-2)a+\frac{n(n-1)}{2}\cos(n-4)a+\ldots+\frac{n(n-1)}{2}\cos(-n+4)a+n\cos(-n+2)a+\cos(-na).$$

Since, in general, $\cos m=\cos(-m)$, the terms of the second member may be

added two and two; but two cases must be distinguished, according as n is even or odd. In the first case, the number of the terms in the second member is odd and the middle term stands alone; this term is

$$\frac{n(n-1)(n-2)\ldots\left\{n-\left(\frac{n}{2}-1\right)\right\}}{1.\,2.\,3\ldots\frac{n}{2}}$$

$$\cos(n-n)a=\frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1.\,2.\,3\ldots\frac{n}{2}}.$$

In the second case, all the terms are doubled, and, if we begin the series with $\cos na+\cos(-na)=2\cos na$, the last term will be

$$2\frac{n(n-1)(n-2)\ldots\left\{n-\left(\frac{n-1}{2}-1\right)\right\}}{1.\,2.\,3\ldots\frac{n-1}{2}}$$

$$\cos[n-(n-1)]a=.$$

$$2\frac{n(n-1)(n-2)\ldots\frac{n+3}{2}}{1.\,2.\,3\ldots\frac{n-1}{2}}\cos a.$$

In the same way may be found the value of $(2\sin a)^n$. We have $\overline{\mathrm{NB}}=\overline{\mathrm{NK}}+\overline{\mathrm{KB}}$, $\overline{\mathrm{NG}}=\overline{\mathrm{NK}}+\overline{\mathrm{KG}}$; but $\overline{\mathrm{NG}}=-\overline{\mathrm{NB}}$ and $\overline{\mathrm{KG}}=\overline{\mathrm{KB}}^{-1}$; whence $2\overline{\mathrm{NB}}=\overline{\mathrm{KB}}-\overline{\mathrm{KB}}^{-1}$, and

$$(2\overline{\mathrm{NB}})^n=\overline{\mathrm{KB}}^n-n.\overline{\mathrm{KB}}^{n-2}+\frac{n(n-1)}{2}.$$

$$\mathrm{KB}^{n-4}-\ldots\pm\frac{n(n-1)}{2}.\overline{\mathrm{KB}}^{-n+4}\mp$$

$$n.\mathrm{KB}^{-n+2}\pm\overline{\mathrm{KB}}^{-n}=(\sim 2\sin a)^n.$$

The upper and lower signs correspond respectively to the cases in which n is even or odd. Examine first the former. $(\sim 2\sin a)^n$ is then of the prime order, and the medial terms in the development of the second member may be neglected; whence

$$\pm(2\sin a)^n=\cos na-n\cos(n-2)a+$$

$$\frac{n(n-1)}{2}\cos(n-4)a-\ldots+\frac{n(n-1)}{2}$$

$$n\cos(-n+4)a-n\cos(-n+2)a+\cos(-na).$$

In the first member we take the plus sign when n is of the form $4m$, and the minus sign when n is of the form $4m+2$. The middle term, which is

$$\frac{n(n-1\ldots\left(\frac{n}{2}+1\right)}{1.\ 2.\ 3\ .\ \ .\frac{n}{2}},$$

as in the formula for the cosine, is not doubled.

In the second case, $(\sim 2\sin a)^n$ is of a medial order. Hence the prime terms of the second member must be dropped, which gives, after dividing the equations by ~ 1,

$$\pm(2\sin a)^n=\sin na-n\sin(n-2)a+\frac{n(n-1)}{2}$$

$$\sin(n-4)a-\ldots\frac{n(n-1)}{2}\sin(-n+4)a$$

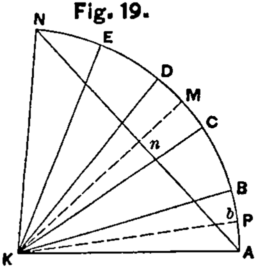
$$+n\sin(-n+2)a-\sin(-na).$$

The $+$ and $-$ signs correspond respectively to the cases in which n is of the form $4m+1$ and $4m+3$. Here all the terms are equal two and two; for, in general, $\sin m=-\sin(-m)$, and the number of terms is even. Uniting, therefore, as above, the equal terms, the series reduces to $\frac{n+1}{2}$ terms, the last of which is

$$\frac{n(n-1)\,(n-2)\,\dots\,\frac{n+3}{2}}{1.\,2.\,3\,\dots\,\frac{n-1}{2}}\sin a.$$

25. Suppose the arc AN divided into n equal parts AB, BC, ... , EN (Fig. 19).

Fig. 19.



Draw AN and AB and through their middle points, n and b, the radii KM, KP. Then

$$\overline{KB}+\overline{KC}+\overline{KD}+\dots+\overline{KN}=$$
$$\cos a\sim\sin a+\cos 2a\sim\sin 2a+\cos 3a\sim\sin 3a+\dots+\cos na\sim\sin na=C\sim S,$$

where

$$C=\cos a+\cos 2a+\cos 3a+\dots+\cos na,$$
$$S=\sin a+\sin 2a+\sin 3a+\dots+\sin na.$$

Let $\overline{KB}=u$, $\overline{KC}=u^2$, ... , $\overline{KN}=u^n$; whence

$$\overline{KB}+\overline{KC}+\dots+\overline{KN}=u+u^2+u^3+\dots+u^n=\frac{u^n-1}{u-1}.\,u=$$
$$\frac{\overline{KN}-\overline{KA}}{\overline{KB}-\overline{KA}}.\,u=\frac{\overline{KN}+\overline{AK}}{\overline{KB}+\overline{AK}}.\,u=$$
$$\frac{\overline{AN}}{\overline{AB}}.\,u=\frac{\overline{nN}}{\overline{bB}}.\,u.$$

But (No. 12, § 4)

$$\overline{nN}=\sim\sin\frac{na}{2}\times\overline{KM}=\sim\sin\frac{na}{2}.\,u^{\frac{n}{2}},$$
$$\overline{bB}=\sim\sin\frac{a}{2}\times\overline{KP}=\sim\sin\frac{a}{2}.\,u^{\frac{1}{2}}.$$

Hence

$$C\sim S=\frac{\sim\sin\frac{na}{2}.\,u^{\frac{n}{2}}}{\sim\sin\frac{a}{2}.\,u^{\frac{1}{2}}}.\,u=\frac{\sin\frac{na}{2}}{\sin\frac{a}{2}}.\,u^{\frac{n+1}{2}}$$
$$=\frac{\sin\frac{na}{2}}{\sin\frac{a}{2}}\left(\cos\frac{n+1}{2}a\sim\sin\frac{n+1}{2}a\right).$$

Equating the terms of like order, we have

$$C=\frac{\sin\frac{na}{2}\cdot\cos\frac{n+1}{2}a}{\sin\frac{a}{2}}$$

$$\text{and}\quad S=\frac{\sin\frac{na}{2}\cdot\sin\frac{n+1}{2}a}{\sin\frac{a}{2}}.$$

26. A similar process will lead to the reduction of

$$K=\cos a+\cos(a+b)+\cos(a+2b)+\ldots+\cos(a+nb),$$

$$\Sigma=\sin a+\sin(a+b)+\sin(a+2b)+\ldots+\sin(a+nb).$$

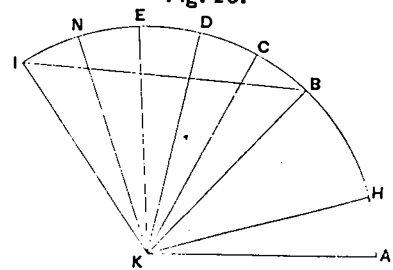
To this end, describe the arcs $AB=a$, BC, CD, ..., $EN=b$ (Fig. 20), the latter being n in number. Make, moreover, $AH=NI=b$, and draw BI and AH. Then if $\overline{KH}=u$, $\overline{KB}=v$, we have

$$\overline{KC}=vu,\ \overline{KD}=vu^2,\ \ldots,\ \overline{KN}=vu^n,\quad \overline{KI}=vu^{n+1}.$$

Therefore

$$K\sim\Sigma=v+vu+vu^2+\ldots+vu^n=\frac{vu^{n+1}-v}{u-1}=\frac{\overline{KI}-\overline{KB}}{\overline{KH}-\overline{KA}}=\frac{\overline{KI}+\overline{BK}}{\overline{KH}+\overline{AK}}$$

Fig. 20.



$$=\frac{\overline{BI}}{\overline{AH}}=\frac{\frac{1}{2}\overline{BI}}{\frac{1}{2}\overline{AH}}=\text{[No. 12, § 6]}$$

$$\frac{\sim\sin\left(\frac{n+1}{2}b\right)\cdot\overline{K}\cdot(\overline{AB}+\frac{1}{2}\overline{BI})}{\sim\sin\frac{1}{2}b\cdot\overline{K}\cdot\frac{1}{2}\overline{AH}}=$$

$$\frac{\sim\sin\left(\frac{n+1}{2}b\right)\cdot\overline{K}\cdot(\overline{AB}+\frac{1}{2}\overline{BN})}{\sim\sin\frac{1}{2}b}=$$

$$\frac{\sin\left(\frac{n+1}{2}b\right)}{\sin\frac{1}{2}b}\left\{\cos\left(a+\frac{bn}{2}\right)\sim\sin\left(a+\frac{bn}{2}\right)\right\},$$

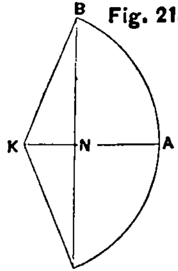
or, equating terms of like order,

$$K=\frac{\sin\left(\frac{n+1}{2}b\right)\cos\left(a+\frac{bn}{2}\right)}{\sin\frac{1}{2}b}$$

and $$\Sigma=\frac{\sin\left(\frac{n+1}{2}b\right)\sin\left(a+\frac{bn}{2}\right)}{\sin\frac{1}{2}b}.$$

27. What precedes is sufficient to show that the method here presented may be employed in trigonometrical researches. It may also be of some use in elementary geometry and algebra, as will now be briefly indicated.

Fig. 21.

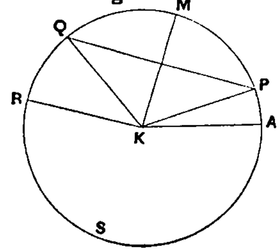


28. Let Fig. 21 be constructed; its simplicity and analogy, with those al-

ready made use of, renders its explanation unnecessary. It follows from the rules of multiplication and addition that $\overline{KB}\times\overline{KG}=\overline{KA}^2$, and $\overline{KB}=\overline{KN}+\overline{NB}$, $\overline{KG}=\overline{KN}+\overline{NG}$. Hence $\overline{KA}^2=(\overline{KN}+\overline{NB})(\overline{KN}+\overline{NG})$. Let KA$=h$, KN$=a$, NB$=NG=b$. Then

$$h^2=(a\sim b)(a+b)=a^2+b^2.$$

Fig. 22.



29. Any directed chord $\overline{PQ}$ (Fig. 22) is of the same order and sign as the radius $\overline{KR}$, drawn in the direction of this chord. Now the angle AKR is equal to $\frac{AP+AQ+\pi}{2}$; for, drawing KM perpendicular to PQ, we have

arc $AR = AP + PM + MR = AP + \frac{1}{2}PQ + \frac{\pi}{2} = AP + \frac{1}{2}(AQ - AP) + \frac{\pi}{2} = \frac{AP + AQ + \pi}{2}$.

The radius $\overline{KR}$ may therefore be expressed (No. 12, § 6) by $K.\overline{\left(\frac{AP+AQ+\pi}{2}\right)}$, an expression indicating the direction of the chord $\overline{PQ}$. On this expression we also remark that the chord $\overline{PQ}$ being indeterminate, the letters P and Q may be interchanged, and for $\overline{QP}$ we should have

$$K.\overline{\left(\frac{AQ+AP+\pi}{2}\right)},$$

an expression identical with the former; for $AP + AQ = AQ + AP$. We should infer from this that $\overline{PQ}$ and $\overline{QP}$ have the same direction, which is, however, not the case, since they are reciprocally positive and negative. To solve this difficulty we observe that the designation of an arc by its two terminal points, as AP, applies to an infinite number of arcs, as $AP + 2n\pi$, n being any integer. In such expressions, then, as the above, of all

these arcs that one should be taken which conforms to the construction followed in the establishment of the general formula. Suppose the point Q to move in the direction QRS until it reaches P, and that, at the same time, P moving in the same direction, reaches Q. The chord $\overline{PQ}$ will then be what was before the chord $\overline{QP}$. The direction of this chord $\overline{PQ}$ will still be

$$K.\overline{\left(\frac{AP+AQ+\pi}{2}\right)};$$

but, in this last expression, the arc AP should be estimated from A, round the entire circumference, plus the arc AP itself, so that this expression really differs from the former by the quantity $\frac{2\pi}{2} = \pi$, as it should.

To avoid all ambiguity, it is sufficient in the general formula

chord $\overline{PQ}$ *having the direction*

$$K.\overline{\left(\frac{AP+AQ+\pi}{2}\right)},$$

to consider the arc AQ greater than the arc AP, following along the circumference in either direction from A to P to determine the arc AP, and then continuing in the same direction till Q is reached. Thus for $\overline{K.\left(\frac{AP+AQ+\pi}{2}\right)}$ we might write $\overline{K.\left(\frac{2AP+PQ+\pi}{2}\right)}$, the arcs AP and PQ being estimated in the same direction. In addition to the foregoing it may be remarked that, if the chord $\overline{PQ}$ is divided at N into any two segments, the part $\overline{NQ}$ has the same direction as $\overline{K.\left(\frac{AP+AQ+\pi}{2}\right)}$, and the part $\overline{NP}$, which relative to $\overline{NQ}$ is negative, has the direction

$$\overline{K.\left(\frac{AP+AQ+\pi}{2}-\pi\right)}=\overline{K.\left(\frac{AP+AQ-\pi}{2}\right)}.$$

Hence, remembering that in general the product of two lines having the directions $\overline{K.FG}$, $\overline{K.HI}$ is in the direction of $\overline{K.(FG+HI)}$, we conclude that the pro-

duct $\overline{NP.NQ}$ will have the direction $\overline{K.(AP+AQ)}$.

30. Take now any four points P,Q,R,S; remembering that in general $\overline{MN}=-\overline{NM}$, we may write

$$\overline{PS}.\overline{QR}+\overline{RS}.\overline{PQ}=\overline{PS}.\overline{QR}+(\overline{RQ}+\overline{QS})(\overline{PS}+\overline{SQ})=\overline{PS}.\overline{QR}+\overline{RQ}.\overline{PS}+\overline{RQ}.\overline{SQ}+\overline{QS}.\overline{PS}+\overline{QS}.\overline{SQ}=\overline{RQ}.\overline{SQ}+\overline{QS}.\overline{PS}+\overline{QS}.\overline{SQ}=\overline{QS}(\overline{QR}+\overline{PS}+\overline{SQ})=\overline{QS}.\overline{PR}.$$

Now, if the points P, Q, R, S are so situated that the three products of the final equation $\overline{PS}.\overline{QR}+\overline{RS}.\overline{PQ}=\overline{QS}.\overline{PR}$ have the same direction, this equation will be true of absolute lines. This condition will be satisfied if the points in question are taken in the order P, Q, R, S, on the circumference, in which case PQ, QR, RS, PS are the sides of a quadrilateral whose diagonals are PR, QS. In fact, these sides and diagonals being so many chords of the circle, we may, by the formula of the preceding article, form the following table, the origin of the arcs A being supposed to immediately precede the point P:

Chords.	Directions.
$\overline{PS}$	$\overline{K.\left(\frac{AP+AS+\pi}{2}\right)}$
$\overline{QR}$	$\overline{K.\left(\frac{AQ+AR+\pi}{2}\right)}$
$\overline{RS}$	$\overline{K.\left(\frac{AR+AS+\pi}{2}\right)}$
$\overline{PQ}$	$\overline{K.\left(\frac{AP+AQ+\pi}{2}\right)}$
$\overline{QS}$	$\overline{K.\left(\frac{AQ+AS+\pi}{2}\right)}$
$\overline{PR}$	$\overline{K.\left(\frac{AP+AR+\pi}{2}\right)}$

and these expressions will be free from ambiguity, because, on account of the supposed order in which A,P,Q,R and S are taken, these six chords are all taken in the same direction.

Hence, in virtue of the principle cited above, the three products $\overline{PS}.\overline{QR}$, $\overline{RS}.\overline{PQ}$, $\overline{QS}.\overline{PR}$ have the same direction, namely, that of

$$\overline{K.\left(\frac{AP+AQ+AR+AS+2\pi}{2}\right)}.$$

Thus, then, for absolute lines, we have PS.QR=RS.PQ+QS.PR. This demonstration, far more simple than the ordinary one founded only on the comparison of similar triangles, is here given only as illustrative of the use of intermedials, of which little has been said.

31. In this last article we propose to show that every polynomial of the form $X^n+aX^{n-1}+bX^{n-2}+\ldots+fX+g$ is decomposable into factors $X+\alpha$ of the first degree. It is to be noticed that $a, b, \ldots g$ are not necessarily reals, as is ordinarily the case.

It is well known that the problem consists in the proof that a quantity can always be found which, substituted for X, will render the polynomial zero, which latter we make $=Y$. Denote by $Y_{(p)}$, $Y_{(p+\rho i)}$ the values of Y obtained by making $X=p$, $X=p+\rho i$, p and i being arbitrary numbers and ρ a directed radius or an indeterminate root of unity. We then have $Y_{(p)} = p^n + ap^{n-1} + bp^{n-2} + \ldots + g$, $Y_{(p+\rho i)} = (p+\rho i)^n + a\,(p+\rho i)^{n-1} + b\,(p+\rho i)^{n-2} + \ldots + g = Y_{(p)} + i\rho\,Q + i^2\rho^2 R + i^3\rho^3 S + \ldots$, Q, R, S being known quantities, dependent on p, n, a, b, c, . . . , and obtained from the development of the powers of $p+\rho i$. If i be supposed infinitely small, the terms containing i^2, i^3, . . . , disappear, and we have simply $Y_{(p+\rho i)} = Y_{(p)} + i\rho Q$. Let $\overline{KP}$ have the direction of $Y_{(p)}$. Assume ρ so that $i\rho Q$ shall have the direction $\overline{PK}$, that is of the same order as $Y_{(p)}$, but opposite in direction; it follows that the magnitude of $Y_{(p+\rho i)}$ will be less than that of $Y_{(p)}$: similarly we may obtain a new value of Y which shall be less than that of $Y_{(p+\rho i)}$, and so on, and finally therefore a value of X for which $Y = o$.

To render the demonstration complete, it must be remarked that the term $i\rho Q$ may become zero. In this case we should retain the succeeding term $i^2\rho^2 R$, or, should this disappear, $i^3\rho^3 S$, and so on. The reasoning remains the same,

because the powers ρ^2, ρ^3, . . . are quantities of the same nature as ρ.

32. The method above explained rests upon two principles of construction, one for the multiplication, the other for the addition of directed lines; and it has been already observed that inasmuch as these principles depend upon inductions which are not securely established, they cannot, as yet, be considered as other than hypotheses, whose acceptance or rejection should depend upon either the consequences which they entail or a more rigorous logic.

We might have dwelt more fully upon the fundamental ideas which lead to these results. We might have indicated, by some comparisons, how certain points, in the theories of Algebra and Geometry, bear upon these principles admitted by induction, whose truth is established rather by the exactness of their consequences than by the logic on which they are founded; but this discussion would have contributed nothing essential to the foregoing, and we confine ourselves to proposing the method of directed lines as an instrument of research, whose use is advantageous in certain cases, because geometric constructions offer, as it were, a picture to the eye which facilitates purely intellectual operations. Moreover, it is always possible to translate the demonstrations founded on this method into ordinary language.

NOTES ON THE

GEOMETRICAL INTERPRETATION

OF

IMAGINARY QUANTITIES

BY

PROF. A. S. HARDY.

NOTES.

The preceding treatise, by Argand, appeared in the year 1806. In Vol. IV., 1813–14, of Gergonne's *Annales de Mathematiques*, appeared an article entitled "New Principles of the Geometry of Position and Geometrical Interpretation of Imaginary Symbols," by J. F. Francais, Professor in the Imperial School of Artillery at Metz, of which the following is an abstract.

The author began by calling attention to the distinction between the magnitude and position of a line, and to the still incomplete state of the geometry of position. He proposed the notation a_α, b_β, ... to represent right lines whose absolute lengths were a, b, the subscript Greek letters denoting the angles made by these lines with any arbitrary axis of reference. Francais used the expression "lines given in magnitude and position," to designate what

Argand called "directed right lines." In the term *ratio* he included the relative position as well as the relative magnitude, four directed lines being in proportion as

$$a_\alpha : b_\beta :: c_\gamma : d_\delta,$$

when $\frac{b}{a}=\frac{d}{c}$, and also $\beta-\alpha=\delta-\gamma$. In such a proportion, the absolute lengths are in geometrical, while the angles made with the axis are in arithmetical progression; and the homologous sides of any two similar complanar figures are in proportion. In conformity with the above definition, the proportion

$$a_\alpha : b_\beta :: b_\beta : c_\gamma$$

involves the equations

$$\frac{b}{a}=\frac{c}{b}, \text{ and } \beta-\alpha=\gamma-\beta,$$

whence $\beta=\frac{1}{2}(\alpha+\gamma)$,

or a mean proportional between the directed lines bisects their included angle. So for the continued proportion

$$a_\alpha : b_\beta : c_\gamma \ldots : l_\lambda : m_\mu,$$

we should have

$$\frac{b}{a}=\frac{c}{b}= \ldots =\frac{m}{l},$$

and $\beta-\alpha=\gamma-\beta=\mu-\lambda$.

He then proposed a second notation. By the former $a_0=a$, and $1_0=1$; therefore $1 : 1_\alpha :: a : a_\alpha$, or $a_\alpha=a.1_\alpha$; so that a directed line might also be represented by the symbol $a.1_\alpha$, a denoting its length and 1_α its position.

Lines parallel to the axis of reference drawn from left (right) to right (left) were distinguished as positive (negative); angles estimated above (below) the axis from right to left were regarded positive (negative). This convention, in connection with the above notation, gave $+1=1_0$, $-1=1_{\pm\pi}$, and therefore

$$+a=a\times(+1)=a.1_0,$$
$$\text{and } -a=a\times(-1)=a.1_{\pm\pi}.$$

And from the known relations

$$+1=e^{0\pi\sqrt{-1}}, \text{ and } -1=e^{\pm\pi\sqrt{-1}},$$

results

$$+a=a\times(+1)=a.e^{0\pi\sqrt{-1}},$$
$$\text{and } -a=a\times(-1)=a.e^{\pm\pi\sqrt{-1}}.$$

He then proceeded to establish four theorems:

I. *In the geometry of position, imaginary quantities of the form* $\pm a\sqrt{-1}$ *represent perpendiculars to the axis of reference, and, conversely, perpendiculars to the axis are imaginaries of this form.*

Demonstration.—The quantity $\pm a\sqrt{-1}$ is a mean proportional between $+a$ and $-a$, that is, between a_0 and a_π; hence by the definition of mean proportional is expressed by $a_{\pm\frac{\pi}{2}}$; or, it is perpendicular to the axis and drawn either above or below it; and we have

$$+a\sqrt{-1}=a_{+\frac{\pi}{2}}, \text{ and } -a\sqrt{-1}=a_{-\frac{\pi}{2}}.$$

. Reciprocally, every perpendicular to this axis is represented, in conformity with the above notation, by $a_{\pm\frac{\pi}{2}}$ and is, therefore, by definition, a mean proportional between a_0 and $a_{\pm\pi}$, or between $+a$ and $-a$. It is therefore an imaginary of the form $\pm a\sqrt{-1}$.

Cor. 1.—As signs of position, $\pm\sqrt{-1}$ is identical with $1_{\pm\frac{\pi}{2}}$.

Cor. 2.—Moreover, since $-1=1_{\pm\pi}=e^{\pm\pi\sqrt{-1}}$, we have also $\pm\sqrt{-1}=1_{\pm\frac{\pi}{2}}=e^{\pm\frac{\pi}{2}\sqrt{-1}}$.

Cor. 3.—So-called imaginary quantities are quite as real as positive or negative quantities, and differ from them only in position, being in fact perpendicular to them.

M. Francais argued that this theory of signs was more consistent than the ordinary one of Cartesian geometry, where, as abscissas and ordinates, two kinds of positive and two kinds of negative quantities were admitted. He contended that having once defined positive and negative quantities, as laid off parallel to the axis of abscissas, it was illogical to admit others not comprised in the definition, and that the common theory was thus faulty in admitting two incompatible principles where one was sufficient.

THEOREM II.—The sign of position $1_\alpha=e^{\alpha\sqrt{-1}}$.

Demonstration.—Let the semi-circumference of a unit circle be divided in the direction of positive arcs into m equal parts, and radii be drawn to the points of division; these radii will form a progression both as to magnitude and position, by definition. The two extremes being $1_0=+1$, and $1_\pi=-1=e^{\pi\sqrt{-1}}$, the means

$1_{\frac{\pi}{m}}, 1_{\frac{2\pi}{m}}, \ldots\ldots 1_{\frac{(m-1)\pi}{m}}$, will be

$$e^{\frac{\pi}{m}\sqrt{-1}},\ e^{\frac{2\pi}{m}\sqrt{-1}}, \ldots\ldots e^{\frac{(m-1)\pi}{m}\sqrt{-1}};$$

or, in general, $1_{\frac{n\pi}{m}} = e^{\frac{n\pi}{m}\sqrt{-1}}$; and as $\frac{n\pi}{m}$ may be any angle whatever, we have finally $1_a = e^{a\sqrt{-1}}$;

From this theorem M. Francais drew the following corollaries:

1. That by taking the logarithms of each member of the last equation, $a\sqrt{-1} = \log(1_a)$; showing that, in the geometry of position, arcs of circles are the logarithms of the corresponding radii, being affected with the sign $\sqrt{-1}$ since they are perpendicular to the axis of reference; explaining also the expression, "imaginary arcs of a circle are logarithms," and giving a rational interpretation of the symbolic equation

$$\frac{\pi}{2}\sqrt{-1} = \log(\sqrt{-1}).$$

2. That since $a_a = a.1_a$, we have also $a_a = a.e^{a\sqrt{-1}}$

3. That since $e^{a\sqrt{-1}} = \cos a + \sin a\sqrt{-1}$,

it follows that $a_a = a\cos a + a\sin a.\sqrt{-1}$, or that to express a directed right line, we must take the sum of its projections in two rectangular co-ordinate axes; each projection being taken with its proper sign of position.

4. That for any such lines we may substitute any number, provided that the sum of the projections of the latter is equal to the sum of the lines themselves: that is, we may write a_a, b_β, ... m_μ for x_ξ, provided we have

$$\text{(A)}\quad x.e^{\xi\sqrt{-1}} = a.e^{a\sqrt{-1}} + b.e^{\beta\sqrt{-1}} + \ldots + m.e^{\mu\sqrt{-1}},$$

$$\text{or (B)}\quad \begin{cases} x\cos\xi = a\cos a + b\cos\beta + \ldots + m\cos\mu, \\ x\sin\xi = a\sin a + b\sin\beta + \ldots + m\sin\mu, \end{cases}$$

and conversely.

If the lines a_a, b_β, x_ξ, etc. form a closed polygon, (B) will be satisfied, and hence for any given line may be substituted a series of others, forming with it a closed polygon; conversely for a series of lines forming an unclosed polygon may be substituted the closing line.

The application of these remarks to the theory of the composition and resolution of forces is evident. On this point M. Francais briefly says, "This theory which has always involved some difficulties is thus reduced to a problem of the Geometry of Position."

THEOREM III.—The sign of position 1_α may also be written $1^{\frac{\alpha}{2\pi}}$, that is to say $1_\alpha = 1^{\frac{\alpha}{2\pi}}$.

Demonstration.—If the unit circle be divided into m equal parts and the radii be drawn, they will form a progression whose extremes are unity. Hence $1_{\frac{2\pi}{m}} = 1^{\frac{1}{m}}$, $1_{\frac{4\pi}{m}} = 1^{\frac{2}{m}}$, ... $1_{\frac{2n\pi}{m}} = 1^{\frac{n}{m}}$. Let then $\frac{2n\pi}{m} = \alpha$; we shall have $\frac{n}{m} = \frac{\alpha}{2\pi}$, and consequently $1_\alpha = 1^{\frac{\alpha}{2\pi}}$.

Cor. 1.—It follows from this theorem: 1°, that the above radii denote the m mth roots of unity; 2°, these roots are all equal, differing only in their positions; 3°, they are all equally real, being represented by lines given both in magnitude and position.

Cor. 2.—Comparing the last two theorems we obtain at once the well-known values of these roots, which may be expressed, in general, by

$$1^{\frac{n}{m}} = e^{\frac{2n\pi}{m}\sqrt{-1}} = \cos\frac{2n\pi}{m} + \sin\frac{2n\pi}{m} \cdot \sqrt{-1}.$$

He then proposes the substitution, for $+$, $-$ and $\pm\sqrt{-1}$, of 1_0, $1_{\pm\pi}$, $1_{\frac{\pi}{2}}$, in connection with the general sign $1_{\pm\alpha}$; an additional advantage over that already suggested being that $+$ and $-$ will indicate addition and subtraction only, and so have but one meaning.

THEOREM IV.—All the roots of an equation of any degree are *real* and may be represented by lines given in magnitude and position.

Demonstration.—It has been shown that every equation of any degree whatever is always decomposable into real factors of the first or second degree, and hence it is sufficient to show that the root of an equation of the second degree can be represented by lines given in magnitude and position. Now the roots of an equation of the second degree, being of the form $x = p \pm \sqrt{q}$, can at once be constructed by the foregoing rules; for, 1°, if q is positive, x will be the sum or difference of two positive or negative quantities, laid off on the axis ; 2°, if q is negative, x will be a right line drawn from the origin, the co-ordinates of whose extremity are p and $\sqrt{q}$.

M. Francais concluded his communication as follows:

Such is a very brief sketch of the new principles on which it seems to me desirable and necessary to found the geometry of position, and which I submit to the judgment of geometers. Being in direct conflict with the commonly received ideas concerning so-called imaginary quantities, I expect they will encounter many objections ; but I dare to think that their thorough examination will show them to be well founded, and that the consequences I have drawn from them, strange as at first sight they may seem, will nevertheless be found in harmony with the most rigorous logic. I ought, moreover, to acknowledge that the germ of these ideas is not my own. I found it in a letter to my late brother from M. Legendre, in which this great geometer gave (as something he himself had derived from another, and purely as a matter of curiosity) the substance of my definitions of proportion and ratio—theorem I. and cor. 3 of theorem II. ; but the latter was a mere suggestion, and only justified by a few applications. To myself, therefore, belongs only the credit due to the

manner in which these principles have been set forth, and their proof, the notation, and the proposal of the symbol $1 \pm a$. I hope that the publicity thus given to my own results may induce the real author of these conceptions to make himself known and to publish his own researches on this subject.

In the same volume (IV. p. 71–73) of the annales which contained this paper from M. Francais, a note was inserted by the editor, Gergonne, to the effect that two years before (1811) in a letter he had written to M. de Maiziére on a communication which the latter had contributed to the first volume, he had suggested that "numerical quantities were perhaps improperly classified in a single series, and that, from their very nature, it seemed as if they should be arranged in a table of double argument, as follows:

. . . .

$$\ldots, -2+2\sqrt{-1},\ -1+2\sqrt{-1},\ +2\sqrt{-1}, +1+2\sqrt{-1}, +2+2\sqrt{-1}, \ldots$$

$$\ldots, -2+\sqrt{-1}, -1+\sqrt{-1}, +\sqrt{-1}, +1+\sqrt{-1},\ +2+\sqrt{-1}, \ldots$$

$$\ldots, -2, \quad -1, \quad \pm 0, \quad +1, \quad +2, \quad \ldots$$
$$\ldots, -2-\sqrt{-1}, -1-\sqrt{-1}, -\sqrt{-1}, +1-\sqrt{-1}, +2-\sqrt{-1}, \ldots$$
$$\ldots, -2-2\sqrt{-1}, -1-2\sqrt{-1}, -2\sqrt{-1}, +1-2\sqrt{-1}, +2-2\sqrt{-1}, \ldots$$
. . . ."

So that, like Francais, he proposed that quantities of the form $n\sqrt{-1}$ should be laid off in a direction perpendicular to that in which the quantity n was measured, and that quantities having other directions should be represented by the sum of their projections on these two. He cites also from a letter of M. de Maiziére the following: "What I have advanced on imaginary quantities is quite novel, ... and I am sure you have already recognized its exactness," and again: "This will cease to be a paradox when I have proved that imaginaries of the second degree, and therefore of all degrees, are no more imaginary than negative quantities or imaginaries of the first degree, and that as regards the

former we are exactly in the same position as were the Algebraists of the seventeenth century with respect to the latter." M. Gergonne disclaims any intention of depriving either Argand or Francais of the credit due them, but simply called attention to the fact that, after all, these conceptions were not so strange as would seem, since several had entertained them, and in closing he remarks that M. Francais' paper may be summarized in the following proposition:

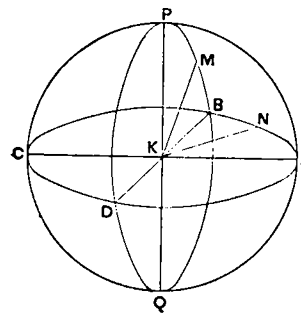
"When, we seek a determinate but unknown length which is supposed to lie in a certain direction along a given line from a given point, while it really lies in the opposite direction, we obtain a negative expression; and if this length is not on the line at all, the expression will appear under an imaginary form."

M. Francais' paper called forth a second article from M. Argand, which appeared in Vol. IV., p. 133–147 of the Annales, wherein he called attention to his previous publication, and claims to have been the person to whom Legendre referred in his letter, he having submitted his first treatise to Legendre's examination. This second paper is, in the main, a restatement of the views advanced in the first; but in it he abandoned the use of the signs $\sim$ and $\dashv\!\vdash$, and returned to that of $\pm\sqrt{-1}$. He also added some further remarks, which are interesting as showing how he attempted to extend his theory to tri-dimensional space, and of which the following is a translation:

Let (Fig. 23) $\overline{KA}=+1$, $\overline{KC}=-1$, $\overline{KB}=+\sqrt{-1}$, $\overline{KD}=-\sqrt{-1}$; any other radius $\overline{KN}$, in the same plane, will be of the form $p+q\sqrt{-1}$; and, conversely, every expression of this form will denote a directed line of this plane.

Draw now from the center K a perpendicular KP=KA to the plane. How shall this directed line be designated? Is it wholly independent of $\overline{KA}$ and $\overline{KB}$, or can it be referred analytically to the prime unit $\overline{KA}$, as are $\overline{KB}$, $\overline{KC}$? Guided by analogy it would seem that, taking the entire circumference as the unit angle, a directed radius making an angle a with $\overline{KA}$ would be expressed, from the principles already laid down, by 1^a; but this expression would be troublesome when a is a fraction, because it would then have more than one value. This objection would be met by the adoption of M. Francais' notation, 1_a; we should thus have $\overline{KA}=1_0$, $\overline{KB}=1_{\frac{1}{4}}$, $\overline{KC}=1_{\frac{1}{2}}$, $\overline{KD}=1_{\frac{3}{4}}$. We have considered angles reckoned

Fig. 23



from A above and below as positive and negative. Now, if we apply to the angles the rule we have adopted for lines, we should be led to regard imaginary angles as laid off in a direction perpendicular to that which corresponds to real angles. Suppose the semi-circumference ABC to revolve about AC, the point B describing the circle BPDQ; since we already have angle $\overline{AKB}=+\frac{1}{4}=\frac{1}{4}.(+1)$,

angle $\overline{AKD}=-\frac{1}{4}=\frac{1}{4}.(-1)$,

we may write angle $\overline{AKP}=\frac{1}{4}\sqrt{-1}=\frac{1}{4}.1_{\frac{1}{4}}$; whence we conclude that

$$KP=1_{\frac{1}{4}\,1_{\frac{1}{4}}}=1_{\frac{1}{4}\sqrt{-1}}=1^{\frac{1}{4}\sqrt{-1}}=$$
$$(1^{\frac{1}{4}})^{\sqrt{-1}}=(\sqrt{-1})^{\sqrt{-1}}.$$

This would seem to be the analytical expression required.

If on the circle BPD we take the point M, so that BKM$=\mu$, we shall have, in like manner, angle $\overline{AKM}=\frac{1}{4}(\cos\mu+\sqrt{-1}\sin\mu)$, and writing for brevity $\cos\mu+\sqrt{-1}\sin\mu=\rho$, $\overline{KM}=1_{\frac{1}{4}\rho}=1.^{\frac{1}{4}\rho}=(1^{\frac{1}{4}})^{\rho}=(\sqrt{-1})^{\cos\mu+\sqrt{-1}\sin\mu}$ will be the general expression for all radii perpendicular to the primitive radius $\overline{KA}$.

Let us now seek an expression for $\overline{BKP}$. On the circumference ABC, the angles estimated from B in either direction are positive and negative, and real, and the plane BKP is perpendicular to their direction; it would thus seem that the angle $\overline{BKP}$, like $\overline{AKP}$, $=\frac{1}{4}\sqrt{-1}$, and that this should in like manner be true for any angle $\overline{NKP}$, N being on the circumference ABCD; but that this conclusion is erroneous is evident from the fact that when N and C coincide, we should have $\overline{CKP}=\frac{1}{4}\sqrt{-1}$, whereas this angle is evidently $-\overline{AKP}=-\frac{1}{4}\sqrt{-1}$. To avoid this difficulty, observe that having

adopted a direction for $+1$, there are an infinity of lines perpendicular to it, among which one is arbitrarily chosen as that of $\sqrt{-1}$. The general expression for every unity taken in one of these directions is, as we have just seen,

$$1_{\frac{1}{4}\rho}=1^{\frac{1}{4}\rho}=(\sqrt{-1})^{\rho}=(\sqrt{-1})^{\cos\mu+\sqrt{-1}\sin\mu}.$$

Conceive at the point A, an infinite number of directions perpendicular to the circumference at that point; one of these will be that of $\overline{KP}$; namely that one we have taken to construct the positive imaginary angles $+a\sqrt{-1}$; that is, for this case we have taken $\rho=1=\overline{KA}$. So, at C, the direction parallel to $\overline{KP}$ gave negative imaginary angles $-a\sqrt{-1}$; that is, we have made $\rho=-1=\overline{KB}$. Hence, with respect to the direction from B parallel to $\overline{KP}$, analogy would lead us to make $\rho=\sqrt{-1}=\overline{KB}$. Thus the expression for $\overline{BKP}$ will be $\frac{1}{4}(\sqrt{-1})^{\sqrt{-1}}$.

We will not further enlarge on these suggestions, and observe only in closing that the expressions a, a_b, a_{b_c}, which designate lines considered in reference to one, two and three dimensions, are only the first terms of a series which can be indefinitely extended.

If the above ideas are admissable, the question so often raised, as to whether every function can be reduced to the form $p+q\sqrt{-1}$, would be answered in the negative; and $\overline{KP}=(\sqrt{-1})^{(\sqrt{-1})}$ would offer the simplest example of a quantity irreducible to this form, and as heterogeneous with respect to $\sqrt{-1}$ as is the latter with respect to $+1$.

It is true there are demonstrations going to show that the form $(a+b\sqrt{-1})^{m+n\sqrt{-1}}$ can always be reduced to the form $p+q\sqrt{-1}$; but we may be permitted to remark that those which make use of series are not conclusive so long as it is not proved that p and q are finite. Indeed it often happens in analysis that a series, which, from its very nature can only be true for real quantities, assumes an infinite value, or rather form, when it is made to represent an imaginary quantity; and in like manner it is presumable that a series composed of terms of the form $p+q\sqrt{-1}$ or a_b can become infinite if it is to express a quantity of the order a_{b_c}. As for those demonstrations which employ logarithms, they also seem somewhat obscure, because we have as yet no definite conceptions of imaginary logarithms. It is also necessary to ascertain whether the same logarithm may not belong at the same time to several quantities of different orders; a, a_b, a_{b_c}. Moreover the several values resulting from the radicals of the proposed expression is another source of ambiguity, so that one may succeed in rigorously reducing $(a+b\sqrt{-1})^{m+n\sqrt{-1}}$ to the form $p+q\sqrt{-1}$ without its being necessarily true that this expression has no other values of the order a_{b_c} irreducible to this form.

Before this second paper of Argand's had come to the notice of Francais, the latter also had endeavored to extend the new theory of imaginaries to tri-dimensional space. In the fourth Vol. of Gergonne's Annales a letter appeared from Francais, from which the following is an extract:

According to my previous definition, positive and negative angles are taken in the same plane, which for brevity I shall designate as the plane xy. It would then seem natural to suppose that imaginary angles are situated in planes perpendicular to xy, and this supposition would be justified by analogy alone; but its legitimacy may be shown as follows: the angle $+\beta\sqrt{-1}$ is a mean proportional, both as to magnitude and position, between $+\beta$ and $-\beta$; it is therefore situated with respect to the angle $+\beta$ as is the angle $-\beta$ with respect to it, which can only be so long as the plane of the angle $\pm\beta\sqrt{-1}$ bisects the angle of the planes $+\beta$ and $-\beta$. Now these planes coincide; therefore the plane of $\pm\beta\sqrt{-1}$ is perpendicular to the plane xy. Conversely, since every plane perpendicular to xy bisects the angles between the planes of the positive and negative angles, every angle β, in such a plane, may be considered a mean proportional in magnitude and position between $+\beta$ and $-\beta$; hence its value, in respect to both magnitude and position, is $\pm\beta\sqrt{-1}$.

From the above, and my 2nd and 3d theorems, it follows that $1_{\beta\sqrt{-1}}=e^{(\beta\sqrt{-1})\sqrt{-1}}=e^{-\beta}$

$$=1^{\frac{\beta\sqrt{-1}}{2\pi}}=\cos(\beta\sqrt{-1})+\sqrt{-1}\sin(\beta\sqrt{-1}).$$

Lambert's *hyperbolic sine and cosine* are thus reduced to the theory of circular arcs, Naperian logarithms, and roots of unity.

It further follows that

$$1_\alpha\cdot 1_{\beta\sqrt{-1}}=e^{\alpha\sqrt{-1}}e^{(\beta\sqrt{-1})\sqrt{-1}}=e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}$$
$$=1_{\alpha+\beta\sqrt{-1}}$$
$$=e^{\alpha\sqrt{-1}}[\cos(\beta\sqrt{-1})+\sqrt{-1}\sin(\beta\sqrt{-1})]$$
$$=\cos\alpha\cos(\beta\sqrt{-1})+\sqrt{-1}\sin\alpha\cos(\beta\sqrt{-1})$$
$$+\sqrt{-1}.e^{\alpha\sqrt{-1}}\sin(\beta\sqrt{-1}).$$

Whence

$$a_{\alpha+\beta\sqrt{-1}}=a\cos\alpha\cos(\beta\sqrt{-1})+\sqrt{-1}$$
$$a\sin\alpha\cos(\beta\sqrt{-1})+\sqrt{-1}\,.\,ae^{\alpha\sqrt{-1}}\sin(\beta\sqrt{-1}).$$

Hence the projections of a on the three coordinate axes, or rather its three components, will be

$$a\cos\alpha\cos(\beta\sqrt{-1}),\quad \sqrt{-1}.a\sin\alpha\cos(\beta\sqrt{-1}),$$
$$\sqrt{-1}.a_\alpha\sin(\beta\sqrt{-1}).$$

These, Monsieur, are the results I have reached; but I confess I am not yet satisfied

with them. I desire to suppress wholly the old imaginary notation, as I have done for geometry of two dimensions; that is, for the latter I have reduced oblique lines of the form $A+B\sqrt{-1}$ to that of a_α, where a denotes the absolute length of the line, and α the angle it makes with the axis of reference. In tri-dimensional geometry, I desire to express the position of any line by a_{α_A}, a denoting the absolute length α the above angle and A the angle made by the plane of a with xy; but as yet all my efforts in that direction have proved unsuccessful. I trust some one more skillful than myself may succeed in filling up this gap. At all events, I am confident that the true method of extending our theory of imaginaries to tri-dimensional geometry consists in the consideration of imaginary angles.

In a postscript to this letter, Francais acknowledges the receipt of Argand's memoir, and that to the latter belongs the credit of the discovery of the geometrical representation of imaginaries.

He then adds:

In starting from the same principle we have reached different results. I have said above that I have not succeeded in reducing the expression for the position of any right line in space to the form a_{a_A}. The reasons for my failure are these: I attempted to make, from analogy

$$a_A = a.e^{A\sqrt{-1}} = a(\cos A + \sqrt{-1}\sin A) \text{ whence}$$

$$1_{a_A} = \left(e^{a\sqrt{-1}}\right)^{e^{A\sqrt{-1}}} = (\cos a + \sqrt{-1}\sin a)^{\cos A + \sqrt{-1}\sin A},$$

which, when $a=\frac{1}{2}\pi$, $A=\frac{1}{2}\pi$, gives

$$1^{\frac{1}{2}\pi}.1_{\frac{1}{2}\pi} = (\sqrt{-1})^{\sqrt{-1}},$$

which agrees with the result of M. Argand. But, developing the general case, we have

$$1_{a_A} = \left(e^{a\sqrt{-1}}\right)^{e^{A\sqrt{-1}}} = e^{(a.e^{A\sqrt{-1}})\sqrt{-1}} = e^{(a\cos A + \sqrt{-1}\,a\sin A)\sqrt{-1}} = e^{\sqrt{-1}.a\cos A}.e^{(\sqrt{-1}a\sin A)\sqrt{-1}}$$

$$= [\cos(a\cos A) + \sqrt{-1}\sin(a\cos A)] \times [\cos(\sqrt{-1}.a\sin A) + \sqrt{-1}\sin(\sqrt{-1}.a\sin A)] =$$
$$\cos(a\cos A)\cos(\sqrt{-1}.a\sin A) + \sqrt{-1}\sin(a\cos A)\cos(\sqrt{-1}.a\sin A) + \sqrt{-1}.e^{\sqrt{-1}.a\cos A}.\sin(\sqrt{-1}.a\sin A),$$

an expression which, on account of its doubly transcendental character, would seem inadmissable. On comparing it with

$$1_{\lambda+\mu\sqrt{-1}} = \cos\lambda\cos(\mu\sqrt{-1}) + \sqrt{-1}\sin\lambda\cos(\mu\sqrt{-1}) + \sqrt{-1}.e^{\lambda\sqrt{-1}}\sin(\mu\sqrt{-1})$$

I rejected it altogether, because the angles a and A are easily found in terms of μ and λ by spherical trigonometry. In fact we have

$$\cos\lambda\cos(\mu\sqrt{-1}) = \cos a,$$
$$\sin\lambda\cos(\mu\sqrt{-1}) = \sin a\cos(A\sqrt{-1}),$$
$$\sin(\mu\sqrt{-1}) = \sin a\sin(A\sqrt{-1});$$

whence

$$\cos\lambda = \frac{\cos a}{\sqrt{1-\sin^2 a\sin^2(A\sqrt{-1})}},$$

$$\sin\lambda = \frac{\sin a\cos(A\sqrt{-1})}{\sqrt{1-\sin^2 a\sin^2(A\sqrt{-1})}}.$$

And therefore

$$1_{a_A} = [\cos a + \sqrt{-1}\sin a\cos(A\sqrt{-1})] \times \left\{1 + \frac{\sin a\sin(A\sqrt{-1})}{\sqrt{1-\sin^2 a\sin^2(A\sqrt{-1})}}\sqrt{-1}\right\}.$$

From this it seems to me clear that a_A cannot be determined as a_a was, and that the supposed analogy between angles and lines does not exist.

You must have remarked, Monsieur, that M. Argand does not prove my proposition $a_a = a(\cos a + \sqrt{-1}\sin a)$, and that this fundamental equality is, with him, simply a supposition justified only by a few examples.

On this remark M. Gergonne very justly observes that no demonstration was needed, inasmuch as Argand had *defined* the sum of directed lines as a certain composition of motions, "a very natural extension of the ordinary definition of Algebra."

M. Francais concludes:

I do not quite see why M. Argand (No. 12), in writing $2\pi=1$, should introduce a new unit, rendering, it seems to me, the rest of his paper obscure. Finally, I should be loth to admit the correctness of his assertion that

$$(c\sqrt{-1})^{d\sqrt{-1}}$$

is irreducible to the form $A+B\sqrt{-1}$. In fact, we have

$$c\sqrt{-1}=e^{\log(c\sqrt{-1})}=e^{\log c+\log\sqrt{-1}}$$
$$=e^{\log c+\frac{1}{2}\pi\sqrt{-1}}=e^{\log c}\,e^{\frac{1}{2}\pi\sqrt{-1}};$$

therefore,

$$(c\sqrt{-1})^{d\sqrt{-1}}=e^{(d\log c)\sqrt{-1}}e^{-\frac{1}{2}d\pi}=e^{-\frac{1}{2}d\pi}$$
$$[\cos(d\log c)+\sqrt{-1}\sin(d\log c)],$$

which is certainly of the form $A+B\sqrt{-1}$. I, therefore, think myself correct in regarding the expression $(c\sqrt{-1})^{d\sqrt{-1}}$, which he assigns to the third dimension, as simply a conjecture open to serious objection.

On Nov. 13th, 1813, M. Servois addressed a letter to Gergonne, which is especially interesting as bearing upon the extension of Argand's theory to space of three dimensions. He objected first to Francais' proof of his first theorem. This proposition, that $\pm a\sqrt{-1}$ is a mean

proportional in magnitude and position between $+a$ and $-a$, he claimed to consist of two, one of which, viz: that $\pm a\sqrt{-1}$, was a mean proportional as to position is not evident, and is indeed precisely what is to be proved. To this criticism Gergonne replied that, although Servois thinks it evident that $\pm a\sqrt{-1}$ is a mean as regards *magnitude*, between $+a$ and $-a$, it seemed to him difficult to see how such an expression, which, with its signs, is a *negation* of *magnitude*, could be a mean between two *reals*; that as regards magnitude, the mean could only be a; but, taking position into account, the mean must also be conceived under this new aspect, and is for this very reason a mean in position as well as magnitude, so that the interpretation of $\pm a\sqrt{-1}$ is reduced to the selection of a line which is situated with reference to $+a$ as $-a$ is to it.

Servois objected, secondly, that the new theory was not only founded merely on analogy, but was not even justified *a posteriori* by its applications. Emphasizing Argand's remark that it consisted in the use of a special notation, he characterized it as "a sort of geometric mask, superadded to analytic forms whose direct use was more simple and expeditious." For example, he says:

. . . take Argand's first application, where he proposes to develope $\sin(a+b)$ and $\cos(a+b)$. From the general formula $e^{a\sqrt{-1}} = \cos a + \sqrt{-1}\sin a$, I obtain $e^{(a+b)\sqrt{-1}} = \cos(a+b) + \sqrt{-1}\sin(a+b)$, and thence $e^{(a+b)\sqrt{-1}} = e^{a\sqrt{-1}}\ e^{b\sqrt{-1}} = (\cos a + \sqrt{-1}\sin a)\ (\cos b + \sqrt{-1}\sin b)$, or $e^{(a+b)\sqrt{-1}} = (\cos a\cos b - \sin a\sin b) + \sqrt{-1}\,(\sin a\ \cos b + \cos a\sin b)$; equating these two values of $e^{(a+b)\sqrt{-1}}$, and subsequently the real and imaginary parts separately, we have $\cos(a+b) = \cos a\cos b - \sin a\sin b$, $\sin(a+b) = \sin a\cos b + \cos a\sin b$. All the other geometrical applications are easily made in the same manner. They may be found in various works, and especially in "A Purely Algebraic Theory of Imaginary Quantities," by M. Suremain-de-Missery (Paris, 1801). The single application to algebra (close of Argand's treatise) seems to me quite unsatisfactory. I do not think it sufficient to find values for x which render the polynomial of less and less value; it is necessary, besides this, that the law

of decrease should necessarily render it *zero*; and that it should be such that zero is not, so to speak, the asymptote of the polynomial.

After citing Euler's proof that

$$(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{1}{2}\pi},$$

in reply to Argand's assertion that this expression was irreducible to the form $p + q\sqrt{-1}$, he raises two other objections, which are important and given in full.

Accustomed to designate the position of a point in a plane by an angle and radius vector, geometers have certainly not been ignorant of the consequences of M. Francais' definition. . . . . But, content with distinguishing between the magnitude and position of a right line in a plane, they had not yet formed, from these two simple ideas, a single complex one, or rather they had not yet created a new *etre geometrique*, uniting at once both the ideas of magnitude and position. The length of a right line and its position, *i. e.* the angle it makes with a fixed axis, are two quantities, which we may term homogeneous; now, how can they be so combined as to form this new entity called a directed line? It seems to me this problem is not yet satisfactorily solved. If a is the length of the line and α the arc of the unit circle which measures the angle it makes with a fixed axis, undoubtedly we may in general represent the line by $\phi\,(a,\,\alpha)$, and the function ϕ must be determined by the essential condition it is to satisfy. Thus 1°, evidently $\phi(o,a)=+a$ must correspond to $\alpha=o$, $\alpha=2\pi,\ .\ .\ .\ .\ ,\ \alpha=2n\pi$, and $\phi(a,\alpha)=-a$ to $\alpha=\pi$, $\alpha=3\pi,\ .\ .\ .\ .\ ,\ \alpha=(2n+1)\pi$; 2°, also, evidently from $\phi(a,\alpha)=\phi(b,\beta)$ we must have $a=b$, $\alpha=\beta$. But 3°, does it follow from the proportion $\frac{\phi(a,\alpha)}{\phi(b,\beta)}=\frac{\phi\,(c,\gamma)}{\phi\,(d,\delta)}$, as M. Francais says, that we must have $\frac{a}{b}=\frac{c}{d}$ and $\alpha-\beta=\gamma-\delta$? I do not see that this necessarily follows from the conception of ϕ. The very meaning of this ratio $\frac{\phi\,(a,\alpha)}{\phi\,(b,\beta)}$ is quite obscure. What indeed is meant by doubling, trebling, etc., a directed line? *A priori* this is not intelligible. M. Francais seems to have been aware of this difficulty, inasmuch as he speaks of the *sum* of directed lines only as a consequence of his first two theorems. Still, I do not object to admitting this condition as an essential characteristic of ϕ; but in that case the complete definition of a directed line will be a definition *nominis non rei*, or, in other words, *directed line* will be the name of a certain analytic function of the length and direction of a right line. From this it unfortunately follows that we

are no longer constructing imaginaries, but simply reducing them to the same analytic form. However, let us see what this function is. It is, in the first place, clear that the expression $\phi(a,\alpha)=a.e^{\alpha\sqrt{-1}}$ satisfies the three foregoing conditions. In fact, we have 1° $\phi(a,o)=a.e^{o\sqrt{-1}}=a$, $\phi(a,\pi)=a.e^{\pi\sqrt{-1}}=a(\cos\pi+\sqrt{-1}\sin\pi)=-a$; 2° the equation $\phi(a,\alpha)=\phi(b,\beta)$ becomes $a.e^{\alpha\sqrt{-1}}=b.e^{\beta\sqrt{-1}}$, or, passing to logarithms, equating, and returning to numbers, $a=b$, $\alpha=\beta$; 3° the above proportion, by similar transformations becomes $\frac{a}{b}=\frac{c}{d}$ and $\alpha-\beta=\gamma-\delta$. But is this form $a.e^{\alpha\sqrt{-1}}$ the only one which satisfies these three conditions? I think not, and it seems to me evident that they will be equally true if we substitute an arbitrary coefficient for the imaginary $\sqrt{-1}$. So that the form $a.e^{\alpha\sqrt{-1}}$ will, in my opinion, only be a special case of the analytic expression for a directed line, in its conventional signification. Are there any other conditions which follow from this signification? To this question no answer is made, nor do I either see any.

Again, 4° the table of double argument which you (Gergonne) propose, as applied to a plane supposed to be so divided into points or *infinitesimal* squares that each square corresponds to a number which would be its *index*, would very properly indicate the length and position of the radii vectores which revolve about the point or central square corresponding to ± 0; and it is quite remarkable that if we designate the length of a radius vector by a, and the angle it makes with the real line ..., -1, $\pm 0, +1$, ... by α, the rectangular co-ordinates of its *extremity remote from the origin* by x, y, the real line being the axis of x, the point would be determined by $x+y\sqrt{-1}$, and consequently, since $x = a\cos\alpha$, $y = a\sin$ by a, $a.e^{\alpha\sqrt{-1}}$. Thus we have a new *geometrical interpretation* of the function $a.e^{\alpha\sqrt{-1}}$ which, it seems to me, is of more value than that of MM. Argand and Francais; but certainly we should not thereby conclude that this was a new method of constructing, *geometrically*, imaginary quantities, for the above indices presuppose them. However this may be, it is clear that your ingenious tabular arrangement of numerical magnitudes may be regarded as a central slice (*tranche centrale*) of a table of triple argument representing points and lines in tri-dimensional space. You would doubtless give to each term a tri-nomial form; but what would be the co-efficient of the third term? For my part I cannot tell. Analogy would seem to indicate that the tri-nominal should be of the form $p\cos\alpha+q\cos\beta+r\cos\gamma$, α, β and γ being the angles made by a right line with three rectangular axes, and that we should have $(p\cos\alpha+q\cos\beta+r\cos\gamma)(p'\cos\alpha+q'\cos\beta+r'\cos\gamma)=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. The values of p,q,r,p',q',r' satisfying this condition would be *absurd*; but would they be imaginaries, reducible to the general form $A+B\sqrt{-1}$?

On this letter Hamilton remarks in his *Lectures on Quaternions*, (Preface, p. 57), "The six *non-reals* which thus Servois with remarkable sagacity *foresaw*, without being able to *determine* them, may now be identified with the then unknown symbols $+i$, $+j$, $+k$, $-i$, $-j$, $-k$ of the quaternion theory;" and it may here be interesting to quote (*North British Review*, 1866), from a letter of Hamilton on the discovery of these symbols:

Oct. 15, '58.

"P. S.—To-morrow will be the fifteenth birthday of the Quaternions. They started into life, or light, full grown, on the 16th of Oct., 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge, which my boys have since called the Quaternion Bridge. That is to say, I then and there felt the galvanic circuit of thought to *close;* and the sparks which fell from it were the *fundamental equations between i, j, k; exactly such* as I have used them ever since. I pulled out, on the spot, a pocketbook, which still exists, and made an entry, on which, *at the very moment,* I felt that it might be worth my while to expend the labor of at least ten (or it might be fifteen) years to come. But then, it is fair to say that this was because I felt a *problem* to have been at that moment *solved*—an intellectual want *relieved*—which had *haunted* me for at least *fifteen years before. Less than an hour* elapsed before I had asked and obtained leave of the Council of the Royal Irish Academy, of which society I was at that time president, to *read,* at the *next general meeting,* a *paper* on Quaternions, which I accordingly *did* on Nov. 13th, 1843."

It is also proper here to add a disclaimer from Gergonne as to any thought of the extension of his table to tridimensional space, until after the appearance of Argand's and Francais' papers; and that even then he saw no way by which to effect that result.

The above letter from M. Servois called forth a reply from Francais (Annales, Vol. IV., p. 364–367), and a third paper from Argand (Annales, Vol. V., p 197–209). In the former, Francais sustains Gergonne, who had already said that Servois asked too much of the new theory, demanding rigorous demonstrations of that which, as in the early history of negative quantities or the calculus, was perceived by a sort of instinct, the proofs of whose fundamental principles the earlier writers were not in a state to produce. He then adds a few examples of the facility with which one might pass from the proposed to the ordinary notation.

The equation of a triangle whose base coincides with the axis of reference is $a_\alpha + b_{-\beta} = c$, whence

$$a \cos \alpha + b \cos \beta = c, \text{ and } a \sin \alpha - b \sin \beta = o,$$

or, taking the sum and difference of the squares

$$a^2 + b^2 + 2ab \cos(\alpha + \beta) = c^2,$$

$$a^2 \cos 2\alpha + b^2 \cos 2\beta + 2ab \cos(\alpha - \beta) = c^2.$$

The equation of the circle referred to the center is $a_\phi = x + y\sqrt{-1}$, whence

$$a\cos\phi = x,\ a\sin\phi = y,\ x^2 + y^2 = a^2.$$

The equation of a circle referred to a diameter is $\rho_\phi + \sigma_{\frac{1}{2}\pi-\phi} = 2a$, whence

$$\rho\cos\phi + \sigma\sin\phi = 2a,\ \rho\sin\phi - \sigma\cos\phi = o,$$
$$\rho^2 = 2a\rho\cos\phi,\ x^2 + y^2 = 2ax.$$

The equation of an ellipse referred to the focus is $\rho_\phi + (2a-\rho)_\Psi = 2e$, whence

$$\rho\cos\phi + (2a-\rho)\cos\Psi = 2e,$$
$$\rho\sin\phi + (2a-\rho)\sin\Psi = o,\ \rho = \frac{a^2 - e^2}{a - e\cos\phi}.$$

The reply of Argand is appended.

The new theory of imaginaries, already referred to several times in this publication, has two distinct and independent objects; it seeks, first, to render intelligible certain expressions whose presence in analysis has been inevitable, but which have not yet been referred to any known evaluable quantity; and, second, it presents a method, or a particular notation which employs geometric symbols concurrently with the ordinary algebraic signs. Hence, from this double point of view, two questions arise: Has it been rigorously shown that $\sqrt{-1}$ represents a line perpendicular to those denoted by $+1$ and -1? Can the notation of directed lines furnish, in certain cases, demonstrations and solutions preferable either for

their simplicity or brevity, etc., to those which they are intended to replace?

The first of these will, perhaps, always be open to discussion so long as we seek to establish the meaning of $\sqrt{-1}$ by analogy, from the commonly received ideas on positive and negative quantities and their ratios. Negative quantities have been and are still the subject of discussion; it will, therefore, be all the easier to raise objections to the new theory of imaginaries. But this difficulty will vanish if, with M. Francais, we *define* what is meant by a ratio of magnitude and position between two lines. Indeed, the relation between two such lines may be conceived of with all necessary precision. Whether this relation be called *ratio* or something else, it may always be made the subject of exact reasoning, and its consequences, in analysis and geometry, of which M. Francais and myself have given some examples, may be traced. The only remaining question, then, is whether it is proper to designate this relation as a *ratio* or *proportion*, words which already possess, in analysis, a determinate and fixed meaning. Now, this is permissible, because the new meaning is an *extension*, not a *contradiction*, of the old one. The latter is so generalized that the ordinary meaning becomes, so to speak, a particular case of the new one. There is then, no question here of *demonstration*.

Thus, for the analyst who first wrote $a^{-n}=\frac{1}{a^n}$, this equation was a definition of negative exponents, not a proposition proved or to be proved. All that it was incumbent upon him to show was that this definition was only a generalization of that of positive exponents, the only ones before known, and so for fractional, irrational and imaginary exponents. It has been said that Euler proved $(\sqrt{-1})^{\sqrt{-1}}=e^{-\frac{1}{2}\pi}$. The word *proved* may be exact if we mean that this equation is derived from $e^{x\sqrt{-1}}=\cos x+\sqrt{-1}\sin x$, which is readily shown to be the case; but it is not so as regards this latter; for to show that a certain expression has a definite value, implies the previous definition of the expression. But is there any definition of imaginary exponents antedating the so called demonstration of Euler? It seems not. When Euler sought to evaluate $e^{x\sqrt{-1}}$, he naturally resorted to the theorem $e^z=1+\frac{z}{1}+\frac{z^2}{1.2}+\ .\ .\ .\ .$ previously demonstrated for all *real* values of z. By making $z=x\sqrt{-1}$ he found

$$e^{x\sqrt{-1}}=1+\frac{x\sqrt{-1}}{1}-\frac{x^2}{1.2}-\ .\ .\ .\ .\ ;$$

Thence he concluded, not that $e^{x\sqrt{-1}}=\cos x+\sqrt{-1}\sin x$, but that, if the expression $e^{x\sqrt{-1}}$ was defined as representing a quantity equal to

$\cos x+\sqrt{-1}\sin x$, we should thereby bring both real and imaginary exponents under the same law. Here again, then, we have the extension of a principle, not the demonstration of a theorem.

It is also by an extension of principles that I was led to regard $(\sqrt{-1})^{(\sqrt{-1})}$ as representing a perpendicular to the plane ± 1, $\pm\sqrt{-1}$. The two results conflict, and I certainly have not insisted upon my own; I only wish to observe that MM. Francais and Servois have attacked it from considerations which are after all of the very nature of those on which I relied to establish it.

But if the above perpendicular cannot be expressed by $(\sqrt{-1})^{\sqrt{-1}}$, how then shall it be represented? Or, rather, can any expression be found, whose adoption, as the representative of the perpendicular, shall bring all directed lines whatever under a common law, as is already the case for every line of the plane ± 1, $\pm\sqrt{-1}$? This is a question which must be of interest to geometers, at least to those who admit the new theory. To return to the original question, I observe that whether $\sqrt{-1}$ does or does not represent the perpendicular on ± 1 must depend upon the meaning of the word *ratio*; for it is agreed by all that $+1:\sqrt{-1}::\sqrt{-1}:-1$ or that $\frac{\sqrt{-1}}{+1}=\frac{-1}{\sqrt{-1}}$.

So that M. Servois' objection to Francais' proof of his first theorem, viz: "That it is not proved that $\pm a\sqrt{-1}$ is a mean, as to position, between $+a$ and $-a$" is equivalent to the assertion that the word *ratio* has no reference to position. In its usual acceptation, this is true; and on the other hand, it may be said that, in the conception of a ratio between quantities with different signs, the signs must be regarded. In the new meaning, direction *and* magnitude make up the idea of ratio. It is thus seen to be a question of words, decided by the exact definition given by Francais, which is an extension of the usual one.

The second point under discussion is more important. Doubtless no truth is reached by the notation of directed lines which cannot be attained by ordinary methods; but which method is the simplest? This question is, I think, worthy of examination. It is to the influence of methods and notations on the progress of the science, that modern mathematics owes its superiority. So that when anything new of this kind appears, we may at least examine it in this respect. Since the publication of the new theory, M. Servois alone has expressed an opinion on this point, and his opinion is not favorable to the new notation. Analytic formulæ seem to him more simple and expeditious. I would, however,

claim for my method a more careful examination. I admit that it is novel, and that the mental operations it requires, although quite simple, demand some familiarity in order that they may be performed with the ease which follows practice in the ordinary operations of Algebra. Some of the theorems I have proved seem to me easier than the corresponding purely analytic processes. This is, perhaps, an author's illusion, and I will not insist upon it: but I claim with more confidence the superiority of the method of directed lines for the demonstration of the Algebraic Theorem: "Every polynomial $x^n+ax^{n-1}+$ . . . . is decomposable into factors of the first or second degree." I feel it necessary to resume this demonstration, not only to reply to the objections of M. Servois, but also to show more fully how easily it is derived from the new principles. The importance and difficulty of this theorem, which has tasked the skill of the best geometers will, I think, excuse, in the eyes of the reader, some repetition. The demonstrations previously given may, I believe, be classified under two heads: Those of the first-class depend on certain metaphysical principles relating to the transformation of functions, which are doubtless true in themselves, but which, properly speaking, are not susceptible of rigorous proof. They are a sort of *axioms* whose truth cannot be appreciated unless we already grasp the *spirit* of Algebraic analysis; whereas to admit the truth of a *theorem*, it is sufficient to know the principles of this analysis; that is, to understand its definitions and language. Hence demonstrations of this kind have been frequently attacked. The *Receuil*, in which these remarks appear, offers several examples, and the appearance of such discussions is an indication of the fact that such reasoning is not above reproach.

In other cases the proposition to be established is approached directly, by showing that there is always at least one quantity of the form $a+b\sqrt{-1}$, which, when substituted for x, renders the polynomial zero; that is to say that this polynomial may always be resolved into factors of the first or second degree. This is the method of Lagrange. This great geometer has shown that the previous methods of d'Alembert, Euler, Foncenex, etc., are inadequate (Résolution des equations numériques. Notes IX. and X). Some of them resorted to series, others to auxiliary equations; but they did not prove, as they should have done, that the co-efficients of these series and equations were always real. These geometers admit implicitly the principle that "if a problem involving an unknown quantity can be resolved in n ways, it must lead to an equation of the *nth* degree." Lagrange himself regards this legitimate, although he does not use it in the above-cited demonstrations. Now, may it not be said that this principle, probably true as it is, is not demonstrated, and belongs to that class of axioms above referred to? Especially would it seem as if this principle, which in theory is among the first to be demonstrated, was out of place, dependent as it is upon no little familiarity with the practice of the science. This remark is not a mere quibble, which, as regards conceptions deserving the respect of all geometers, would be as out of place as it is useless, but is made simply to show the difficulty in the way of a satistory treatment of this subject.

It would appear from the above that a demonstration at once simple, direct and rigorous is worthy the attention of geometers. I shall, therefore, resume that of my previous paper; but, to avoid all ambiguity, shall free it from any consideration of vanishing quantities. It will be convenient to restate briefly the first principles of the theory of directed lines.

Having taken $\overline{KA}$ as the direction of positive quantities, the opposite direction $\overline{AK}$, will be, as usual, that of negative quantities. Drawing the perpendicular BKD through K, one of the directions $\overline{KB}$, $\overline{KD}$, the former say, will correspond to imaginaries of the form $+a\sqrt{-1}$, the latter to those of the form $-a\sqrt{-1}$. The line drawn above the letters indicates that direction is considered, and, when we are only concerned with length, is suppressed. Assuming arbitrarily the points F, G, H, ... , P, Q, we have $\overline{FG}+\overline{GH}+ \ldots +\overline{PQ}=\overline{FQ}$. This is the law of addition. If, between four lines, there exists the relation $\frac{AB}{CD}=\frac{EF}{GH}$, and, in addition, the angle between $\overline{AB}$, $\overline{CD}$ is equal to that between $\overline{EF}$, $\overline{GH}$, these lines are said to be *in proportion*. Hence the law of multiplication; for a product is merely a fourth term in a proportion whose first term is unity.

It is to be observed that these two rules are independent of any opinion one may have on the new theory. If it is desirable that $\sqrt{-1}$, a symbol to which the practice of Algebra continually gives rise, and which, sometimes called absurd, has yet never given absurd results, if it is desirable I repeat that this symbol should remain meaningless, while still not being zero, this will give rise to no difficulty. Directed lines will only be the *symbols* of numbers of the form $a+b\sqrt{-1}$. The above rules will be none the less true, but instead of deducing them *a priori* from purely metaphysical considerations, the

first will depend on a simple construction. The second will be an immediate consequence of the formulæ $\sin(a+b)=\sin a \cos b+ \ldots$: and therefore the use of these rules may give demonstrations entirely satisfactory.

Directed lines will then be symbols of the numbers $a+b\sqrt{-1}$. Like them they are susceptible of increase, decrease, multiplication, division, etc.; they will, as it were, correspond throughout, function for function, and, in a word, *represent* them completely. Hence, from this point of view, concrete quantities will represent abstract numbers; but conversely abstract numbers cannot represent concrete quantities. In what follows, the accents and subscripts are used to indicate the absolute magnitude of the quantities to which they are affixed; thus, if $a=m+n\sqrt{-1}$, m and n being real, it is understood that a_1 or $a'=\sqrt{m^2+n^2}$. Let then

$$y_x=x^n+ax^{n-1}+bx^{n-2}+ \quad . \quad . \quad . \quad . \quad +fx+g$$

be the proposed polynomial, n being a whole number; a, b, ... f, g may be of the form $m+n\sqrt{-1}$. We are to prove that we may always find a quantity such that, substituted for x, $y_x=o$. The polynomial may be constructed for any value of x by the preceding rules. Taking K as the initial point, P as the final one, $\overline{KP}$ will express this polynomial, and it is to be shown that x may be so determined as to cause P to coincide with K. Now, if among all the possible values of x, there is no one of which this is true, the line $\overline{KP}$ cannot become null; and of all the values of $\overline{KP}$ there will necessarily be one smaller than the rest. Designate this minimum value of x by z; then $y'(z+i) < y'z$ cannot be true, whatever the value of i. Now, developing, we have

$$
(A)\quad \left\{\begin{array}{l}
y(z+i) = yz + \\
\qquad [nz^{n-1} + (n-1)az^{n-2} + \ldots\ldots + f\,]i + \\
\left\{\frac{n}{1}\cdot\frac{n-1}{2}z^{n-2} + \ldots\ldots\right\} i^2 + \ldots\ldots \\
\qquad\qquad + (nz+a)i^{n-1} + i^n.
\end{array}\right.
$$

As the co-efficients of the several powers of i may become zero, and this is a special case, it is better to replace the above equation by

$$(B)\quad y(z+i) = yz + Ri^r + Si^s + \ldots\ldots + Vi^v + i^n;$$

and so make the solution general; R, S and V not being zero and the exponents $r, s, \ldots, v, n$ being increasing. Observe that if all the coefficients of (A) were zero, the equation would reduce to $y(z+i) = yz + i^n$. Making then $i = \sqrt[n]{-yz}$, we shall have $y(z+i) = o$, and the theorem would be established for this case, which in what follows may therefore be set aside. We shall then suppose that the second member of (B) has at least three terms. With this premise, construct $y'(z+i)$, taking $\overline{KP} = yz$,

$\overline{PA} = Ri^r$, $\overline{AB} = Si^s$, ..., $\overline{FG} = Vi^v$, $\overline{GH} = i^n$; we shall have $y'z = KP$, $R'i_1{}^r = PA$, $S'i_1{}^s = AB$, ... , $V'i_1{}^v = FG$, $i_1{}^n = GH$; for evidently, in general, $p'q' = (pq)'$.

$y_{(z+i)}$ will be represented by the broken or straight line

$$\overline{KPAB\ldots FGH}, \text{ or by } \overline{KH};$$

and it is to be proved that we have $KH < KP$.

Now the quantity i may vary in two ways:

1°. In *direction*; and it is clear that if it varies by an angle a, its power i^r will vary by an angle ra. Let then a be the angle by which $\overline{PA} = Ri^r$ is greater than $KP = yz$. If i is made to vary by the angle $\frac{\pi - a}{r}$, $\overline{PA}$ will vary by the angle $\pi - a$; that is, the direction of $\overline{PA}$ will become opposite to that of $\overline{KP}$; so that the point A will be found on the line PK, prolonged, if necessary, through K.

2°. The direction of i being supposed fixed, we may, in the second place, cause it to vary in *magnitude*; and first, if $PA > KP$, we may diminish i till $PA < KP$, so that A will fall between K and P. Then, if the magnitude of i, so diminished, is not such that $R'i_1{}^r > S'i_1{}^s + \ldots\ldots + V'i_1{}^v + i_1{}^n$, we may, by diminishing it still further, make this inequality true, for the exponents $s, \ldots, v, n$ are all greater than r. Now this inequality is equivalent to PA > AB + . . . + FG + GH; therefore the distance AH will be less than PA, and, consequently, if we describe a circle with A as a center and radius AP, the point H will lie within this circle, and it follows from elementary geometry, that K being on the prolongation of the radius PA, in the direction of the center A, we shall have KH < KP.

To follow this demonstration, I would ask the reader to make the diagram. By the application thereto of the above cited simple fundamental principles, it will be seen that, with the exception of the development (A) which is algebraic, the remainder of the demonstration is made, as it were, at sight, without any mental effort.

It is almost superfluous to dwell upon an objection which might be made to what precedes, namely, that if one undertook to diminish the value of x by the method prescribed for diminishing $y'x$, one might never succeed, because the value of i, in the successive substitutions, might diminish by constantly decreasing quantities. Indeed, the contrary is not proved; but from this it only follows that the above considerations cannot furnish, at least without new developments, an approximative method, and this does not in the least invalidate the demonstration of the theorem.

M. Servois' objection is easily answered. "It seems to me," he says, "that it is not enough to find values of x which render the polynomial constantly less; it is necessary, in addition, that the law of diminution should necessarily reduce the polynomial to zero, or, if I may use the expression, such that zero is not the *asymptote* of the polynomial." It has been proved that we may not only find for y'_x constantly diminishing values, but a value less than any assignable one. If the polynomial cannot be reduced to zero, its least value will then be other than zero, and in this case also the demonstration holds good. The close of M. Servois' sentence would seem to indicate that he makes a distinction between an infinitely small limit and one which is absolutely zero. If such was his meaning he might be answered in the words of M. Gergonne . . . . Doubtless M. Servois' difficulty arises from the equation of the hyperbola $y = \frac{1}{x}$. It is unquestionably true that in this equation, although we may assign to y a value less than any assignable one, y cannot become zero unless x is supposed infinite. But this is not the case in our demonstration; for certainly it is not an infinite value for x which will render the polynomial $y'x$ zero.

Let us now resume the question which has given rise to the above explanations. It may be asked if it is possible to translate what precedes into the ordinary language of analysis. It seems to me quite probable, although it may be difficult in this way to obtain so simple a result. To effect this it would seem necessary to assimilate the notation of imaginaries to that of directed lines, writing, for instance:

$$\sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}+\frac{b}{\sqrt{a^2+b^2}}\sqrt{-1}\right\}$$

for $a+b\sqrt{-1}$;

$\sqrt{a^2+b^2}$ might be called the *modulus* of $a+b\sqrt{-1}$, and would represent the absolute length of the line $a+b\sqrt{-1}$, while the other factor, whose modulus is unity would denote its direction. We should only prove 1° that *the modulus of the sum of several quantities is not greater than the sum of their moduli*, which is equivalent to saying that the line AF is not greater than the sum of the lines AB, BC, ... EF; 2° *that the modulus of the product of several quantities is equal to the product of their moduli.*

The further investigation of the relations between the notations I must leave to those more skillful than myself. If this attempt to obtain a purely analytic demonstration as simple as that derived from the new theory is successful, analysis will be the gainer in thus reaching, by an easy method, a result whose

difficulties were not unworthy the notice of Lagrange himself. If, on the contrary, this attempt should prove unsuccessful, the notation of directed lines will retain an evident advantage over the ordinary one; and in either case the new theory will have rendered some small service to science.

In closing, I may be permitted to make a remark on a note from M. Lacroix (Annales, Vol. IV, p. 367). This learned professor says that the Philosophical Transactions of 1806 contain a memoir from M. Buée on the very subject of which M. Francais and myself have written. Now, it was in this same year that my essay appeared, a pamphlet in which I explained the principles of the new theory, and of which the paper inserted in Vol. IV of the *Annales* (p. 133) is but an extract; it is well known, too, that the publications of academies can appear only after the date which they bear. This is sufficient to prove that if the contribution of M. Buée was wholly his own, as is quite possible, it is also quite certain that I could have had no knowledge of his paper when my treatise appeared.

In the foot notes to the Preface of Hamilton's Lectures on Quaternions, the reader will find full references to the labors of other writers on this subjeet, including Warren (1828), Peacock, Ohm, Mourey (1828), Gauss, Buée (1806), Gompertz (1818), Carnot, Wallis (1685), MacCullagh, Argand, Francais, Servois, Grassmann, DeMorgan, Graves, DeFoncenex, Euler, etc. While giving full credit to the results of his predecessors and co-workers, Hamilton justly claims to be alone the founder of a *system*. Moreover, the fundamental conception of this system was radically different from those entertained by previous writers. In the latter inclined or perpendicular lines to the plus and minus axis were represented by imaginaries, whereas *all* unit lines in space are represented by Hamilton by distinct square roots of negative unity, they being all real. No *one* direction is assumed positive, nor is any system of reference chosen independent of the lines of the construction involved in any special problem.

In addition to the works of Hamilton, Tait and Kelland, may be especially mentioned the *Calcolo dei Quaternioni, Bellavitis, Modena*, 1858, and the original

paper (Memoirs of the Italian Society, 1854), by the same author, which has also been translated from the Italian into French by Laisant (*Exposition de lá Méthode des Equipollences. Paris*, 1874); the *Théorie Elementaire des Quantités Complexes*, by Hoüel, Paris, 1874; the *Fonctions doublement périodiques*, of MM. Briot and Bouquet, and a treatise by Allégret, *Sur le Calcul des Quaternions de M. Hamilton.* Paris, 1862.

As possessing some historic interest may be added, in addition to the works cited in the above-mentioned Preface, Truel, 1786, referred to by Cauchy, Woodhouse (Phil. Trans. 1801), Khun, (Nouveaux Mémoires de Petersburg, Vol. 3), and *Le Calcul Directif*, a series of articles by Transom, in the *Nouvelles Annales de Mathématiques*, 1868.

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